$\pi/2$-Angle Yao Graphs are Spanners

Prosenjit Bose*[1], Mirela Damian**[2], Karim Douïeb*[3], Joseph O'Rourke[4], Ben Seamone[5], Michiel Smid*[6], and Stefanie Wuhrer[7]

[1] School of Computer Science, Carleton University, Ottawa, Canada. jit@scs.carleton.ca.

[2] Department of Computer Science, Villanova University, Villanova, USA. mirela.damian@villanova.edu.

[3] School of Computer Science, Carleton University, Ottawa, Canada. kdouieb@ulb.ac.be.

[4] Department of Computer Science, Smith College, Northampton, USA. orourke@cs.smith.edu.

[5] School of Mathematics and Statistics, Carleton University, Ottawa, Canada. bseamone@connect.carleton.ca.

[6] School of Computer Science, Carleton University, Ottawa, Canada. michiel@scs.carleton.ca.

[7] Institute for Information Technology, National Research Council, Ottawa, Canada. stefanie.wuhrer@nrc-cnrc.gc.ca.

Abstract. We show that the Yao graph Y_4 in the L_2 metric is a spanner with stretch factor $8\sqrt{2}(29 + 23\sqrt{2})$.

1 Introduction

Let V be a finite set of points in the plane and let $G = (V, E)$ be the complete Euclidean graph on V. We will refer to the points in V as *nodes*, to distinguish them from other points in the plane. The *Yao graph* [7] with an integer parameter $k > 0$, denoted Y_k, is defined as follows. Any k equally-separated rays starting at the origin define k cones. Pick a set of arbitrary, but fixed cones. We can now translate the cones to each node $u \in V$. In each cone, pick a shortest edge uv, if there is one, and add to Y_k the directed edge $\overrightarrow{uv}$. Ties are broken arbitrarily. Note that the Yao graph differs from the Θ-graph in how the shortest edge is chosen. While the Yao graph chooses the shortest edge in terms of the Euclidean distance, the Θ-graph chooses the shortest edge as the one that has the shortest distance to u after being projected to the bisector of the cone. Most of the time we ignore the direction of an edge uv; we refer to the directed version $\overrightarrow{uv}$ of uv only when its origin (u) is important and unclear from the context. We will distinguish between Y_k, the Yao graph in the Euclidean L_2 metric, and Y_k^∞, the Yao graph in the L_∞ metric. Unlike Y_k however, in constructing Y_k^∞ ties are broken by always selecting the most counterclockwise edge; the reason for this choice will become clear in Section 2.

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For a given subgraph $H \subseteq G$ and a fixed $t \geq 1$, H is called a *t-spanner* for G if, for any two nodes $u, v \in V$, the shortest path in H from u to v is no longer than t times the length of uv. The value t is called the *dilation* or the *stretch factor* of H. If t is constant, then H is called a *length spanner*, or simply a *spanner*.

The class of graphs Y_k has been much studied. Bose et al. [2] showed that, for $k \geq 9$, Y_k is a spanner with stretch factor $\frac{1}{\cos\frac{2\pi}{k} - \sin\frac{2\pi}{k}}$. In [1] we improve the stretch factor and show that, in fact, Y_k is a spanner for any $k \geq 7$. Recently, Molla [5] showed that Y_2 and Y_3 are not spanners, and that Y_4 is a spanner with stretch factor $4(2+\sqrt{2})$, for the special case when the nodes in V are in convex position (see also [3]). The authors conjectured that Y_4 is a spanner for arbitrary point sets. In this paper, we settle their conjecture and prove that Y_4 is a spanner with stretch factor $8\sqrt{2}(29+23\sqrt{2})$.

The paper is organized as follows. In Section 2, we prove that the graph Y_4^∞ is a spanner with stretch factor 8. In Section 3 we establish several properties for the graph Y_4. Finally, in Section 4, we use the properties of Section 3 to prove that, for every edge ab in Y_4^∞, there exists a path between a and b in Y_4 not much longer than the Euclidean distance between a and b. By combining this with the result of Section 2, it follows that Y_4 is a spanner.

2 Y_4^∞ in the L_∞ Metric

In this section we focus on Y_4^∞, which has a nicer structure compared to Y_4. First we prove that Y_4^∞ is a plane graph. Then we use this property to show that Y_4^∞ is an 8-spanner. To be more precise, we prove that for any two nodes a and b, the graph Y_4^∞ contains a path between a and b whose length (in the L_∞-metric) is at most $8|ab|_\infty$.

We need a few definitions. We say that two edges ab and cd *properly cross* (or *cross*, for short) if they share a point other than an endpoint (a, b, c or d); we say that ab and cd *intersect* if they share a point (either an interior point or an endpoint). Let $Q_1(a)$, $Q_2(a)$, $Q_3(a)$ and $Q_4(a)$ be the four quadrants at a, as in

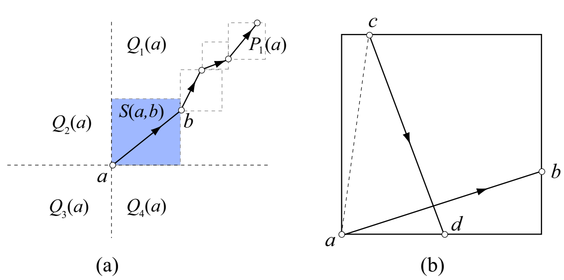

Fig. 1. (a) Definitions: $Q_i(a)$, $P_i(a)$ and $S(a,b)$. (b) Lemma 1: ab and cd cannot cross.

Figure 1a. Let $P_i(a)$ be the path that starts at point a and follows the directed Yao edges in quadrant Q_i. Let $P_i(a,b)$ be the subpath of $P_i(a)$ that starts at a and ends at b. Let $|ab|_\infty$ be the L_∞ distance between a and b. Let $sp(a,b)$ denote a shortest path in Y_4^∞ between a and b. Let $S(a,b)$ denote the open square with corner a whose boundary contains b, and let $\partial S(a,b)$ denote the boundary of $S(a,b)$. These definitions are illustrated in Figure 1a. For a node $a \in V$, let $x(a)$ denote the x-coordinate of a and $y(a)$ denote the y-coordinate of a.

Lemma 1. *Y_4^∞ is a plane graph.*

Proof. The proof is by contradiction. Assume the opposite. Then there are two edges $\overrightarrow{ab}, \overrightarrow{cd} \in Y_4^\infty$ that cross each other. Since $\overrightarrow{ab} \in Y_4^\infty$, $S(a,b)$ must be empty of nodes in V, and similarly for $S(c,d)$. Let j be the intersection point between ab and cd. Then $j \in S(a,b) \cap S(c,d)$, meaning that $S(a,b)$ and $S(c,d)$ must overlap. However, neither square may contain a, b, c or d. It follows that $S(a,b)$ and $S(c,d)$ coincide, meaning that c and d lie on $\partial S(a,b)$ (see Figure 1b). Since cd intersects ab, c and d must lie on opposite sides of ab. Thus either ac or ad lies counterclockwise from ab. Assume without loss of generality that ac lies counterclockwise from ab; the other case is identical. Because $S(a,c)$ coincides with $S(a,b)$, we have that $|ac|_\infty = |ab|_\infty$. In this case however, Y_4^∞ would break the tie between ac and ab by selecting the most counterclockwise edge, which is $\overrightarrow{ac}$. This contradicts that $\overrightarrow{ab} \in Y_4^\infty$. □

Theorem 1. *Y_4^∞ is an 8-spanner in the L_∞ metric space.*

Proof. We show that, for any pair of points $a, b \in V$, $|sp(a,b)|_\infty < 8|ab|_\infty$. The proof is by induction on the pairwise distance between the points in V. Assume without loss of generality that $b \in Q_1(a)$, and $|ab|_\infty = |x(b) - x(a)|$. Consider the case in which ab is a closest pair of points in V (the base case for our induction). If $ab \in Y_4^\infty$, then $|sp(a,b)|_\infty = |ab|_\infty$. Otherwise, there must be $ac \in Y_4^\infty$, with $|ac|_\infty = |ab|_\infty$. But then $|bc|_\infty < |ab|_\infty$ (see Figure 2a), a contradiction.

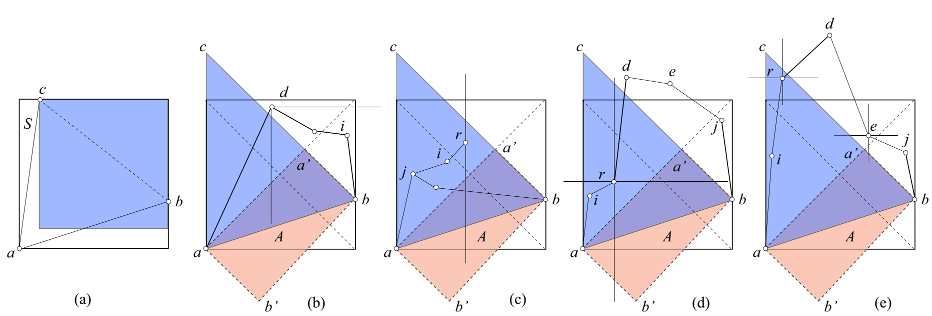

Fig. 2. (a) Base case. (b) $\triangle abc$ empty (c) $\triangle abc$ non-empty, $P_{ar} \cap P_2(b) = \{j\}$ (d) $\triangle abc$ non-empty, $P_{ar} \cap P_2(b) = \emptyset$, e above r (e) $\triangle abc$ non-empty, $P_{ar} \cap P_2(b) = \emptyset$, e below r.

Assume now that the inductive hypothesis holds for all pairs of points closer than $|ab|_\infty$. If $ab \in Y_4^\infty$, then $|sp(a,b)|_\infty = |ab|_\infty$ and the proof is finished. If $ab \notin Y_4^\infty$, then the square $S(a,b)$ must be nonempty.

Let A be the rectangle $ab'ba'$ as in Figure 2b, where ba' and bb' are parallel to the diagonals of S. If A is nonempty, then we can use induction to prove that $|sp(a,b)|_\infty <= 8|ab|_\infty$ as follows. Pick $c \in A$ arbitrary. Then $|ac|_\infty + |cb|_\infty = |x(c) - x(a)| + |x(b) - x(c)| = |ab|_\infty$, and by the inductive hypothesis $sp(a,c) \oplus sp(c,b)$ is a path in Y_4^∞ no longer than $8|ac|_\infty + 8|cb|_\infty = 8|ab|_\infty$; here $\oplus$ represents the concatenation operator. Assume now that A is empty. Let c be at the intersection between the line supporting ba' and the vertical line through a (see Figure 2b). We discuss two cases, depending on whether $\triangle abc$ is empty of points or not.

Case 1: $\triangle abc$ is empty of points. Let $ad \in P_1(a)$. We show that $P_4(d)$ cannot contain an edge crossing ab. Assume the opposite, and let $st \in P_4(d)$ cross ab. Since $\triangle abc$ is empty, s must lie above bc and t below ab, therefore $|st|_\infty \geq |y(s)-y(t)| > |y(s)-y(b)| = |sb|_\infty$, contradicting the fact that $st \in Y_4^\infty$. It follows that $P_4(d)$ and $P_2(b)$ must meet in a point $i \in P_4(d) \cap P_2(b)$ (see Figure 2b). Now note that $|P_4(d,i) \oplus P_2(b,i)|_\infty \leq |x(d)-x(b)| + |y(d)-y(b)| < 2|ab|_\infty$. Thus we have that $|sp(a,b)|_\infty \leq |ad \oplus P_4(d,i) \oplus P_2(b,i)|_\infty < |ab|_\infty + 2|ab|_\infty = 3|ab|_\infty$.

Case 2: $\triangle abc$ is nonempty. In this case, we seek a short path from a to b that does not cross to the underside of ab, to avoid oscillating paths that cross ab arbitrarily many times. Let r be the rightmost point that lies inside $\triangle abc$. Arguments similar to the ones used in Case 1 show that $P_3(r)$ cannot cross ab and therefore it must meet $P_1(a)$ in a point i. Then $P_{ar} = P_1(a,i) \oplus P_3(r,i)$ is a path in Y_4^∞ of length

$$|P_{ar}|_\infty < |x(a) - x(r)| + |y(a) - y(r)| < |ab|_\infty + 2|ab|_\infty = 3|ab|_\infty. \quad (1)$$

The term $2|ab|_\infty$ in the inequality above represents the fact that $|y(a) - y(r)| \leq |y(a) - y(c)| \leq 2|ab|_\infty$. Consider first the simpler situation in which $P_2(b)$ meets P_{ar} in a point $j \in P_2(b) \cap P_{ar}$ (see Figure 2c). Let $P_{ar}(a,j)$ be the subpath of P_{ar} extending between a and j. Then $P_{ar}(a,j) \oplus P_2(b,j)$ is a path in Y_4^∞ from a to b, therefore $|sp(a,b)|_\infty \leq |P_{ar}(a,j) \oplus P_2(b,j)|_\infty < 2|y(j) - y(a)| + |ab|_\infty \leq 5|ab|_\infty$.

Consider now the case when $P_2(b)$ does not intersect P_{ar}. We argue that, in this case, $Q_1(r)$ may not be empty. Assume the opposite. Then no edge $st \in P_2(b)$ may cross $Q_1(r)$. This is because, for any such edge, $|sr|_\infty < |st|_\infty$, contradicting $st \in Y_4^\infty$. This implies that $P_2(b)$ intersects P_{ar}, again a contradiction to our assumption. This establishes that $Q_1(r)$ is nonempty. Let $rd \in P_1(r)$. The fact that $P_2(b)$ does not intersect P_{ar} implies that d lies to the left of b. The fact that r is the rightmost point in $\triangle abc$ implies that d lies outside $\triangle abc$ (see Figure 2d). It also implies that $P_4(d)$ shares no points with $\triangle abc$. This along with arguments similar to the ones used in case 1 show that $P_4(d)$ and $P_2(b)$ meet in a point $j \in P_4(d) \cap P_2(b)$. Thus we have found a path

$$P_{ab} = P_1(a,i) \oplus P_3(r,i) \oplus rd \oplus P_4(d,j) \oplus P_2(b,j) \quad (2)$$

extending from a to b in Y_4^∞. If $|rd|_\infty = |x(d)-x(r)|$, then $|rd|_\infty < |x(b)-x(a)| = |ab|_\infty$, and the path P_{ab} has length

$$|P_{ab}|_\infty \leq 2|y(d) - y(a)| + |ab|_\infty < 7|ab|_\infty. \tag{3}$$

In the above, we used the fact that $|y(d) - y(a)| = |y(d) - y(r)| + |y(r) - y(a)| < |ab|_\infty + 2|ab|_\infty$. Suppose now that

$$|rd|_\infty = |y(d) - y(r)|. \tag{4}$$

In this case, it is unclear whether the path P_{ab} defined by (2) is short, since rd can be arbitrarily long compared to ab. Let e be the clockwise neighbor of d along the path P_{ab} (e and b may coincide). Then e lies below d, and either $de \in P_4(d)$, or $ed \in P_2(e)$ (or both). If e lies above r, or at the same level as r (i.e., $e \in Q_1(r)$, as in Figure 2d), then

$$|y(e) - y(r)| < |y(d) - y(r)| \tag{5}$$

Since $rd \in P_1(r)$ and e is in the same quadrant of r as d, we have $|rd|_\infty \leq |re|_\infty$. This along with inequalities (4) and (5) implies $|re|_\infty > |y(e) - y(r)|$, which in turn implies $|re|_\infty = |x(e) - x(r)| \leq |ab|_\infty$, and so $|rd|_\infty \leq |ab|_\infty$. Then inequality (3) applies here as well, showing that $|P_{ab}|_\infty < 7|ab|_\infty$.

If e lies below r (as in Figure 2e), then

$$|ed|_\infty \geq |y(d) - y(e)| \geq |y(d) - y(r)| = |rd|_\infty. \tag{6}$$

Assume first that $ed \in P_2(e)$, or $|ed|_\infty = |x(e) - x(d)|$. In either case, $|ed|_\infty \leq |er|_\infty < 2|ab|_\infty$. This along with inequality (6) shows that $|rd|_\infty < 2|ab|_\infty$. Substituting this upper bound in (2), we get $|P_{ab}|_\infty \leq 2|y(d) - y(a)| + 2|ab|_\infty < 8|ab|_\infty$. Assume now that $ed \notin P_2(e)$, and $|ed|_\infty = |y(e)-y(d)|$. Then $ee' \in P_2(e)$ cannot go above d (otherwise $|ed|_\infty < |ee'|_\infty$, contradicting $ee' \in P_2(e)$). This along with the fact $de \in P_4(d)$ implies that $P_2(e)$ intersects P_{ar} in a point k. Redefine $P_{ab} = P_{ar}(a, k) \oplus P_2(e, k) \oplus P_4(e, j) \oplus P_2(b, j)$. Then P_{ab} is a path in Y_4^∞ from a to b of length $|P_{ab}| \leq 2|y(r) - y(a)| + |ab|_\infty \leq 5|ab|_\infty$. □

This theorem will be employed in Section 4.

3 Y_4 in the L_2 Metric

In this section we establish basic properties of Y_4. Due to space restrictions, some of these properties are stated without proofs. The proofs can be found in [1]. The ultimate goal of this section is to show that, if two edges in Y_4 cross, there is a short path between their endpoints (Lemma 8). We begin with a few definitions.

Let $Q(a, b)$ denote the infinite quadrant with origin at a that contains b. For a pair of nodes $a, b \in V$, define recursively a directed path $\mathcal{P}(a \to b)$ from a to b in Y_4 as follows. If $a = b$, then $\mathcal{P}(a \to b) = null$. If $a \neq b$, there must exist $\overrightarrow{ac} \in Y_4$ that lies in $Q(a, b)$. In this case, define

$$\mathcal{P}(a \to b) = \overrightarrow{ac} \oplus \mathcal{P}(c \to b).$$

Recall that $\oplus$ represents the concatenation operator. This definition is illustrated in Figure 3a. Fischer et al. [4] show that $\mathcal{P}(a \to b)$ is well defined and lies entirely inside the square centered at b whose boundary contains a.

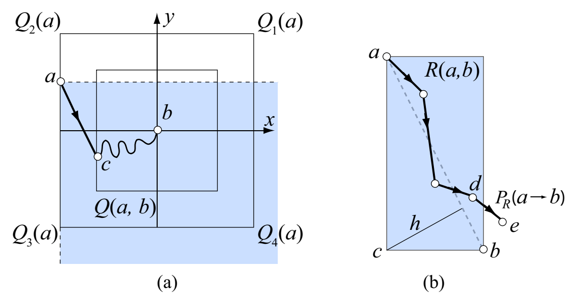

Fig. 3. Definitions. (a) $Q(a, b)$ and $\mathcal{P}(a \to b)$. (b) $\mathcal{P}_R(a \to b)$.

For any node $a \in V$, let $D(a, r)$ denote the open disk centered at a of radius r, and let $\partial D(a, r)$ denote the boundary of $D(a, r)$. Let $D[a, r] = D(a, r) \cup \partial D(a, r)$. For any path P and any pair of nodes $a, b \in P$, let $P[a, b]$ be the subpath of P from a to b. Let $R(a, b)$ be the closed rectangle with diagonal ab.

For a fixed pair of nodes $a, b \in V$, define a path $\mathcal{P}_R(a \to b)$ as follows. Let $e \in V$ be the first node along $\mathcal{P}(a \to b)$ that is not strictly interior to $R(a, b)$. Then $\mathcal{P}_R(a \to b)$ is the subpath of $\mathcal{P}(a \to b)$ that extends between a and e. In other words, $\mathcal{P}_R(a \to b)$ is the path that follows the Y_4 edges pointing towards b, truncated as soon as it reaches b or leaves $R(a, b)$. Formally, $\mathcal{P}_R(a \to b) = \mathcal{P}(a \to b)[a, e]$. This definition is illustrated in Figure 3b. Our proofs will make use of the following two propositions.

Proposition 1. *The sum of the lengths of crossing diagonals of a non-degenerate (necessarily convex) quadrilateral abcd is strictly greater than the sum of the lengths of either pair of opposite sides:*

$$|ac| + |bd| > |ab| + |cd|$$
$$|ac| + |bd| > |bc| + |da|$$

Proposition 2. *For any triangle $\triangle abc$, the following inequalities hold:*

$$|ac|^2 \begin{cases} < |ab|^2 + |bc|^2, & \text{if } \angle abc < \pi/2 \\ = |ab|^2 + |bc|^2, & \text{if } \angle abc = \pi/2 \\ > |ab|^2 + |bc|^2, & \text{if } \angle abc > \pi/2 \end{cases}$$

Lemma 2. *For each pair of nodes $a, b \in V$,*

$$|\mathcal{P}_R(a \to b)| \le |ab|\sqrt{2} \tag{7}$$

Furthermore, each edge of $\mathcal{P}_R(a \to b)$ is no longer than $|ab|$.

Proof. Let c be one of the two corners of $R(a, b)$, other than a and b. Let $\overrightarrow{de} \in \mathcal{P}_R(a \to b)$ be the last edge on $\mathcal{P}_R(a \to b)$, which necessarily intersects $\partial R(a, b)$ (note that it is possible that $e = b$). Refer to Figure 3b. Then $|de| \le |db|$, otherwise $\overrightarrow{de}$ could not be in Y_4. Since db lies in the rectangle with diagonal ab, we have that $|db| \le |ab|$, and similarly for each edge on $\mathcal{P}_R(a \to b)$. This establishes the latter claim of the lemma. For the first claim of the lemma, let $p = \mathcal{P}_R(a \to b)[a, d] \oplus db$. Since $|de| \le |db|$, we have that $|\mathcal{P}_R(a \to b)| \le |p|$. Since p lies entirely inside $R(a, b)$ and consists of edges pointing towards b, we have that p is an xy-monotone path. It follows that $|p| \le |ac| + |cb|$, which is bounded above by $|ab|\sqrt{2}$. □

Lemma 3. *Let $a, b, c, d \in V$ be four disjoint nodes such that $\overrightarrow{ab}, \overrightarrow{cd} \in Y_4$, $b \in Q_i(a)$ and $d \in Q_i(c)$, for some $i \in \{1, 2, 3, 4\}$. Then ab and cd cannot cross.*

The next four lemmas (4–8) each concern a pair of crossing Y_4 edges, culminating (in Lemma 8) in the conclusion that there is a short path in Y_4 between a pair of endpoints of those edges.

Lemma 4. *Let a, b, c and d be four disjoint nodes in V such that $\overrightarrow{ab}, \overrightarrow{cd} \in Y_4$, and ab crosses cd. Then (i) the ratio between the shortest side and the longer diagonal of the quadrilateral $acbd$ is no greater than $1/\sqrt{2}$, and (ii) the shortest side of the quadrilateral $acbd$ is strictly shorter than either diagonal.*

Lemma 5. *Let a, b, c, d be four distinct nodes in V, with $c \in Q_1(a)$, such that (i) $\overrightarrow{ab} \in Q_1(a)$ and $\overrightarrow{cd} \in Q_2(c)$ are in Y_4 and cross each other, and (ii) ad is a shortest side of quadrilateral $acbd$. Then $\mathcal{P}_R(a \to d)$ and $\mathcal{P}_R(d \to a)$ have a nonempty intersection.*

Lemma 6. *Let a, b, c, d be four distinct nodes in V, with $c \in Q_1(a)$, such that (i) $\overrightarrow{ab} \in Q_1(a)$ and $\overrightarrow{cd} \in Q_3(c)$ are in Y_4 and cross each other, and (ii) ad is a shortest side of quadrilateral $acbd$. Then $\mathcal{P}_R(d \to a)$ does not cross ab.*

The next lemma relies on all of Lemmas 2–6.

Lemma 7. *Let $a, b, c, d \in V$ be four distinct nodes such that $\overrightarrow{ab} \in Y_4$ crosses $\overrightarrow{cd} \in Y_4$, and let xy be a shortest side of the quadrilateral $abcd$. Then there exist two paths $\mathcal{P}_x$ and $\mathcal{P}_y$ in Y_4, where $\mathcal{P}_x$ has x as an endpoint and $\mathcal{P}_y$ has y as an endpoint, with the following properties:*

(i) *$\mathcal{P}_x$ and $\mathcal{P}_y$ have a nonempty intersection.*
(ii) $|\mathcal{P}_x| + |\mathcal{P}_y| \le 3\sqrt{2}|xy|$.
(iii) *Each edge on $\mathcal{P}_x \cup \mathcal{P}_y$ is no longer than $|xy|$.*

Proof. Assume without loss of generality that $b \in Q_1(a)$. We discuss the following exhaustive cases:

1. $c \in Q_1(a)$, and $d \in Q_1(c)$. In this case, ab and cd cannot cross each other (by Lemma 3), so this case is finished.

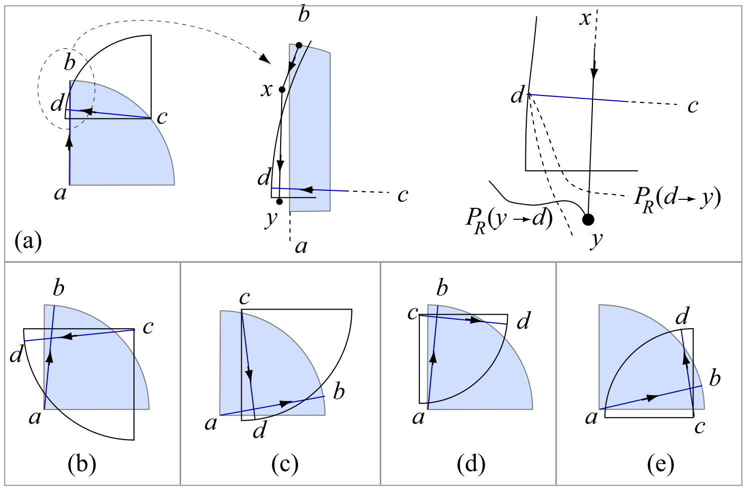

Fig. 4. Lemma 7: (a, b) $c \in Q_1(a)$ (c) $c \in Q_2(a)$ (d) $c \in Q_4(a)$.

2. $c \in Q_1(a)$, and $d \in Q_2(c)$, as in Figure 4a. Since ab crosses cd, $b \in Q_2(c)$. Since $\overrightarrow{ab} \in Y_4$, $|ab| \leq |ac|$. Since $\overrightarrow{cd} \in Y_4$, $|cd| \leq |cb|$. These along with Lemma 4 imply that ad and db are the only candidates for a shortest edge of $acbd$. Assume first that ad is a shortest edge of $acbd$. By Lemma 3, $\mathcal{P}_a = \mathcal{P}_R(a \to d)$ does not cross cd. It follows from Lemma 5 that $\mathcal{P}_a$ and $\mathcal{P}_d = \mathcal{P}_R(d \to a)$ have a nonempty intersection. Furthermore, by Lemma 2, $|\mathcal{P}_a| \leq |ad|\sqrt{2}$ and $|\mathcal{P}_d| \leq |ad|\sqrt{2}$, and no edge on these paths is longer than $|ad|$, proving the lemma true for this case. Consider now the case when db is a shortest edge of $acbd$ (see Figure 4a). Note that d is below b (otherwise, $d \in Q_2(c)$ and $|cd| > |cb|$) and, therefore, $b \in Q_1(d)$). By Lemma 3, $\mathcal{P}_d = \mathcal{P}_R(d \to b)$ does not cross ab. If $\mathcal{P}_b = \mathcal{P}_R(b \to d)$ does not cross cd, then $\mathcal{P}_b$ and $\mathcal{P}_d$ have a nonempty intersection, proving the lemma true for this case. Otherwise, there exists $\overrightarrow{xy} \in \mathcal{P}_R(b \to d)$ that crosses cd (see Figure 4a). Define

$$\mathcal{P}_b = \mathcal{P}_R(b \to d) \oplus \mathcal{P}_R(y \to d)$$
$$\mathcal{P}_d = \mathcal{P}_R(d \to y)$$

By Lemma 3, $\mathcal{P}_R(y \to d)$ does not cross cd. Then $\mathcal{P}_b$ and $\mathcal{P}_d$ must have a nonempty intersection. We now show that $\mathcal{P}_b$ and $\mathcal{P}_d$ satisfy conditions (i) and (iii) of the lemma. Proposition 1 applied on the quadrilateral $xdyc$ tells us that $|xc| + |yd| < |xy| + |cd|$. We also have that $|cx| \geq |cd|$, since $\overrightarrow{cd} \in Y_4$ and x is in the same quadrant of c as d. This along with the inequality above implies $|yd| < |xy|$. Because $xy \in \mathcal{P}_R(b \to d)$, by Lemma 2 we have that $|xy| \leq |bd|$, which along with the previous inequality shows that $|yd| < |bd|$. This along with Lemma 2 shows that condition (iii) of the lemma is satisfied.

Furthermore, $|\mathcal{P}_R(y \to d)| \le |yd|\sqrt{2}$ and $|\mathcal{P}_R(d \to y)| \le |yd|\sqrt{2}$. It follows that $|\mathcal{P}_b| + |\mathcal{P}_d| \le 3\sqrt{2}|bd|$.

3. $c \in Q_1(a)$, and $d \in Q_3(c)$, as in Figure 4b. Then $|ac| \ge \max\{ab, cd\}$, and by Lemma 4 ac is not a shortest edge of $acbd$. The case when bd is a shortest edge of $acbd$ is settled by Lemmas 3 and 2: Lemma 3 tells us that $\mathcal{P}_d = \mathcal{P}_R(d \to b)$ does not cross ab, and $\mathcal{P}_b = \mathcal{P}_R(b \to d)$ does not cross cd. It follows that $\mathcal{P}_d$ and $\mathcal{P}_b$ have a nonempty intersection. Furthermore, Lemma 2 guarantees that $\mathcal{P}_d$ and $\mathcal{P}_b$ satisfy conditions (ii) and (iii) of the lemma. Consider now the case when ad is a shortest edge of $acbd$; the case when bc is shortest is symmetric. By Lemma 6, $\mathcal{P}_R(d \to a)$ does not cross ab. If $\mathcal{P}_R(a \to d)$ does not cross cd, then this case is settled: $\mathcal{P}_d = \mathcal{P}_R(d \to a)$ and $\mathcal{P}_a = \mathcal{P}_R(a \to d)$ satisfy the three conditions of the lemma. Otherwise, let $\overrightarrow{xy} \in \mathcal{P}_R(a \to d)$ be the edge crossing cd. Arguments similar to the ones used in case 1 above show that $\mathcal{P}_a = \mathcal{P}_R(a \to d) \oplus \mathcal{P}_R(y \to d)$ and $\mathcal{P}_d = \mathcal{P}_R(d \to y)$ are two paths that satisfy the conditions of the lemma.
4. $c \in Q_1(a)$, and $d \in Q_4(c)$, as in Figure 4c. Note that a horizontal reflection of Figure 4c, followed by a rotation of $\pi/2$, depicts a case identical to case 1, which has already been settled.
5. $c \in Q_2(a)$, as in Figure 4d. Note that Figure 4d rotated by $\pi/2$ depicts a case identical to case 1, which has already been settled.
6. $c \in Q_3(a)$. Then it must be that $d \in Q_1(c)$, otherwise cd cannot cross ab. By Lemma 3 however, ab and cd may not cross, unless one of them is not in Y_4.
7. $c \in Q_4(a)$, as in Figure 4e. Note that a vertical reflection of Figure 4e depicts a case identical to case 1, so this case is settled as well. □

We are now ready to establish the main lemma of this section, showing that there is a short path between the endpoints of two intersecting edges in Y_4.

Lemma 8. *Let $a, b, c, d \in V$ be four distinct nodes such that $\overrightarrow{ab} \in Y_4$ crosses $\overrightarrow{cd} \in Y_4$, and let xy be a shortest side of the quadrilateral $abcd$. Then Y_4 contains a path $p(x, y)$ connecting x and y, of length $|p(x,y)| \le \frac{6}{\sqrt{2}-1} \cdot |xy|$. Furthermore, no edge on $p(x, y)$ is longer than $|xy|$.*

Proof. Let $\mathcal{P}_x$ and $\mathcal{P}_y$ be the two paths whose existence in Y_4 is guaranteed by Lemma 7. By condition (iii) of Lemma 7, no edge on $\mathcal{P}_x$ and $\mathcal{P}_y$ is longer than $|xy|$. By condition (i) of Lemma 7, $\mathcal{P}_x$ and $\mathcal{P}_y$ have a nonempty intersection. If $\mathcal{P}_x$ and $\mathcal{P}_y$ share a node $u \in V$, then the path $p(x, y) = \mathcal{P}_x[x, u] \oplus \mathcal{P}_y[y, u]$ is a path from x to y in Y_4 no longer than $3\sqrt{2}|xy|$; the length restriction follows from guarantee (ii) of Lemma 7. Otherwise, let $\overrightarrow{a'b'} \in \mathcal{P}_x$ and $\overrightarrow{c'd'} \in \mathcal{P}_y$ be two edges crossing each other. Let $x'y'$ be a shortest side of the quadrilateral $a'c'b'd'$, with $x' \in \mathcal{P}_x$ and $y' \in \mathcal{P}_y$. Lemma 7 tells us that $|a'b'| \le |xy|$ and $|c'd'| \le |xy|$. These along with Lemma 4 imply that $|x'y'| \le |xy|/\sqrt{2}$. This enables us to derive a recursive formula for computing a path $p(x, y) \in Y_4$ as follows:

$$p(x,y) = \begin{cases} x, & \text{if } x = y \\ \mathcal{P}_x[x, x'] \oplus \mathcal{P}_y[y, y'] \oplus p(x', y'), & \text{if } x \ne y \end{cases}$$

Simple induction on the length of xy establishes the claim of the lemma. □

4 Y_4^∞ and Y_4

We prove that every individual edge of Y_4^∞ is spanned by a short path in Y_4. This, along with the result of Theorem 1, establishes that Y_4 is a spanner. Fix an edge $\overrightarrow{xy} \in Y_4^\infty$. Define an edge or a path as *t-short* (with respect to $|xy|$) if its length is within a constant factor t of $|xy|$. In our proof that ab is spanned by a t-short path with respect to $|ab|$ in Y_4, we will make use of the following three statements.

S1 If ab is t-short, then $\mathcal{P}_R(a \rightarrow b)$, and therefore its reverse, $\mathcal{P}_R^{-1}(a \rightarrow b)$, are $t\sqrt{2}$-short by Lemma 2.
S2 If $ab \in Y_4$ is t_1-short and $cd \in Y_4$ is t_2-short, and if ab intersects cd, Lemmas 4 and 8 show that there is a t_3-short path between any two of the endpoints of these edges with $t_3 = t_1 + t_2 + 3(2+\sqrt{2})\max(t_1, t_2)$.
S3 If $p(a, b)$ is a t_1-short path and $p(c, d)$ is a t_2-short path and the two paths intersect, then there is a t_3-short path P between any two of the endpoints of these paths with $t_3 = t_1 + t_2 + 3(2+\sqrt{2})\max(t_1, t_2)$, by **S2**.

Lemma 9. *For any edge $ab \in Y_4^\infty$, there is a path $p(a, b) \in Y_4$ between a and b, of length $|p(a, b)| \leq t|ab|$, for $t = 29 + 23\sqrt{2}$.*

Proof. For the sake of clarity, we only prove here that there is a short path $p(a, b)$ between a and b, and skip the calculations of the actual stretch factor t (which are detailed in the appendix of [1]). We refer to an edge or a path as *short* if its length is within a constant factor of $|ab|$. Assume without loss of generality that $\overrightarrow{ab} \in Y_4^\infty$, and $\overrightarrow{ab} \in Q_1(a)$. If $\overrightarrow{ab} \in Y_4$, then $p(a, b) = ab$ and the proof is finished. So assume the opposite, and let $\overrightarrow{ac} \in Q_1(a)$ be the edge in Y_4; since $Q_1(a)$ is nonempty, $\overrightarrow{ac}$ exists. Because $\overrightarrow{ac} \in Y_4$ and b is in the same quadrant of a as c, we have that

$$\begin{aligned} |ac| &\leq |ab| && \text{(i)} \\ |bc| &\leq |ac|\sqrt{2} && \text{(ii)} \end{aligned} \tag{8}$$

Thus both ac and bc are short. And this in turn implies that $\mathcal{P}_R(b \rightarrow c)$ is short by **S1**. We next focus on $\mathcal{P}_R(b \rightarrow c)$. Let $b' \notin R(b, c)$ be the other endpoint of $\mathcal{P}_R(b \rightarrow c)$. We distinguish three cases.

Case 1: $\mathcal{P}_R(b \rightarrow c)$ and ac intersect. Then by **S3** there is a short path $p(a, b)$ between a and b.

Case 2: $\mathcal{P}_R(b \rightarrow c)$ and ac do not intersect, and $\mathcal{P}_R(b' \rightarrow a)$ and ab do not intersect (see Figure 5b). Note that because b' is the endpoint of the short path $\mathcal{P}_R(b \rightarrow c)$, the triangle inequality on $\triangle abb'$ implies that ab' is short, and therefore $\mathcal{P}_R(b' \rightarrow a)$ is short. We consider two cases:

(i) $\mathcal{P}_R(b' \rightarrow a)$ intersects ac. Then by **S3** there is a short path $p(a, b')$. So

$$p(a, b) = p(a, b') \oplus \mathcal{P}_R^{-1}(b \rightarrow c)$$

is short.

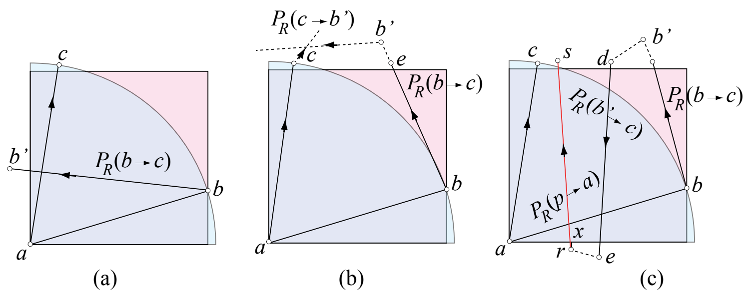

Fig. 5. Lemma 9: (a) Case 1: $\mathcal{P}_R(b \to c)$ and ac have a nonempty intersection. (b) Case 2: $\mathcal{P}_R(b' \to a)$ and ab have an empty intersection. (c) Case 3: $\mathcal{P}_R(b' \to a)$ and ab have a non-empty intersection.

(ii) $\mathcal{P}_R(b' \to a)$ does not intersect ac. Then $\mathcal{P}_R(c \to b')$ must intersect $\mathcal{P}_R(b \to c) \oplus \mathcal{P}_R(b' \to a)$. Next we establish that $b'c$ is short. Let $\overrightarrow{eb'}$ be the last edge of $\mathcal{P}_R(b \to c)$, and so incident to b' (note that e and b may coincide). Because $\mathcal{P}_R(b \to c)$ does not intersect ac, b' and c are in the same quadrant for e. It follows that $|eb'| \leq |ec|$ and $\angle b'ec < \pi/2$. These along with Proposition 2 for $\triangle b'ec$ imply that $|b'c|^2 < |b'e|^2 + |ec|^2 \leq 2|ec|^2 < 2|bc|^2$ (this latter inequality uses the fact that $\angle bec > \pi/2$, which implies that $|ec| < |bc|$). It follows that

$$|b'c| \leq |bc|\sqrt{2} \;\leq\; 2|ac| \qquad \text{(by (8)ii)} \tag{9}$$

Thus $b'c$ is short, and by **S1** we have that $\mathcal{P}_R(c \to b')$ is short. Since $\mathcal{P}_R(c \to b')$ intersects the short path $\mathcal{P}_R(b \to c) \oplus \mathcal{P}_R(b' \to a)$, there is by **S3** a short path $p(c, b)$, and so

$$p(a, b) = ac \oplus p(c, b)$$

is short.

Case 3: $\mathcal{P}_R(b \to c)$ and ac do not intersect, and $\mathcal{P}_R(b' \to a)$ intersects ab (see Figure 5c). If $\mathcal{P}_R(b' \to a)$ intersects ab at a, then $p(a, b) = \mathcal{P}_R(b \to c) \oplus \mathcal{P}_R(b' \to a)$ is short. So assume otherwise, in which case there is an edge $\overrightarrow{de} \in \mathcal{P}_R(b' \to a)$ that crosses ab. Then $d \in Q_1(a)$, $e \in Q_3(a) \cup Q_4(a)$, and e and a are in the same quadrant for d. Note however that e cannot lie in $Q_3(a)$, since in that case $\angle dae > \pi/2$, which would imply $|de| > |da|$, which in turn would imply $\overrightarrow{de} \notin Y_4$. So it must be that $e \in Q_4(a)$.

Next we show that $\mathcal{P}_R(e \to a)$ does not cross ab. Assume the opposite, and let $\overrightarrow{rs} \in \mathcal{P}_R(e \to a)$ cross ab. Then $r \in Q_4(a)$, $s \in Q_1(a) \cup Q_2(a)$, and s and a are in the same quadrant for r. Arguments similar to the ones above show that $s \notin Q_2(a)$, so s must lie in $Q_1(a)$. Let d be the L_∞ distance from a to b. Let x be the projection of r on the horizontal line through a. Then

$$|rs| \geq |rx| + d \geq |rx| + |xa| > |ra| \quad \text{(by the triangle inequality)}$$

Because a and s are in the same quadrant for r, the inequality above contradicts $\overrightarrow{rs} \in Y_4$.

We have established that $\mathcal{P}_R(e \to a)$ does not cross ab. Then $\mathcal{P}_R(a \to e)$ must intersect $\mathcal{P}_R(e \to a) \oplus de$. Note that de is short because it is in the short path $\mathcal{P}_R(b' \to a)$. Thus ae is short, and so $\mathcal{P}_R(a \to e)$ and $\mathcal{P}_R(e \to a)$ are short. Thus we have two intersecting short paths, and so by **S3** there is a short path $p(a, e)$. Then

$$p(a, b) = p(a, e) \oplus \mathcal{P}_R^{-1}(b' \to a) \oplus \mathcal{P}_R^{-1}(b \to c)$$

is short. Straightforward calculations show that, in each of these cases, the stretch factor for $p(a, b)$ does not exceed $29 + 23\sqrt{2}$. □

Our main result follows immediately from Theorem 1 and Lemma 9:

Theorem 2. *Y_4 is a t-spanner, for $t \geq 8\sqrt{2}(29 + 23\sqrt{2})$.*

5 Conclusion

Our results settle a long-standing open problem, asking whether Y_4 is a spanner or not. We answer this question positively, and establish a loose stretch factor of $8\sqrt{2}(29 + 23\sqrt{2})$. Experimental results, however, indicate a stretch factor of the order $1 + \sqrt{2}$, a factor of 200 smaller. Finding tighter stretch factors for both Y_4^∞ and Y_4 remain interesting open problems. Establishing whether Y_5 and Y_6 are spanners or not is also open.

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