Rotationally Monotone Polygons*

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Abstract

A generalization of monotonicity is introduced. An n-vertex polygon P is *rotationally monotone w.r.t. a point* r if there exists a partitioning of the boundary of P into exactly two polygonal chains, s.t. one chain can be rotated clockwise around r and the other chain can be rotated counterclockwise around r with neither chain intersecting the interior of the polygon. We present the following two results: (1) Given P and a center of rotation r in the plane, we determine in $O(n)$ time whether P is rotationally monotone w.r.t. r. (2) We can find all the points in the plane from which P is rotationally monotone in $O(n)$ time for convex polygons and in $O(n^2)$ time for simple polygons. Both algorithms are worst-case optimal.

1 Introduction

Determining whether a polygon has certain properties, such as convexity, monotonicity, or star-shapedness, is a well-studied problem in computational geometry. This problem is not only important from a theoretical point of view, but also from a practical point of view. For surveys and application areas of classes of polygons, the reader is referred to the Handbook of Discrete and Computational Geometry [6, Chapter 23].

A polygon P is monotone in direction $\vec{d}$ if the intersection of P and any line in direction $\vec{d}$ is a convex set. Preparata and Supowit [8] determine in $O(n)$ time whether an n-vertex polygon is monotone. Rosenbloom and Rappaport [9] determine in $O(n)$ time whether a polygon P can be partitioned into exactly two monotone chains, where the two chains are monotone with different directions. Furthermore, they determine in $O(n \log n)$ time whether P can be decomposed into two monotone chains by cutting the boundary along a straight line. Dean et al. [4] introduce pseudo-star-shaped polygons. A polygon P is pseudo-star-shaped if there exists a point r, such that the interior of P is visible from r if one can see through single edges. ElGindy and Toussaint [5] consider radially monotone polygons. A polygon P is radially monotone if there exists a point r, such that every infinite half line emanating from r intersects P in a connected component. Note that the definitions of radially monotone and pseudo-star-shaped are equivalent.

Toussaint [10] introduces another generalization of monotonicity. A polyhedron is weakly-monotonic if there exists a direction $\vec{d}$ s.t. the intersection of the polyhedron and any plane with normal $\vec{d}$ forms a simply-connected set. Bose and van Kreveld [1] give an algorithm to determine in $O(n \log n)$ time whether a simple n-vertex polyhedron is weakly-monotonic.

We introduce a new generalization of monotone polygons. A polygon P is *rotationally monotone* w.r.t. a point r in the plane if the boundary of P can be decomposed into exactly two polygonal chains, s.t. one chain can be rotated in clockwise orientation around r and the other chain can be rotated in counterclockwise orientation around r without either chain penetrating the interior of P. Two problems are addressed. First, given a center of rotation r in the plane, determine whether P is rotationally monotone w.r.t. r. We present a linear time algorithm to solve this problem. Second, an algorithm is presented to find all the points r in the plane, s.t. P is rotationally monotone w.r.t. r. The algorithm's running time for convex polygons is linear and for simple polygons is quadratic. We show that both algorithms are optimal in the worst case.

The notion of rotationally monotone polygons has a direct application to *clamshell casting*. Assume that we wish to manufacture an object modeled by a simple polygon P with n vertices. Let the boundary of P be the *cast* of P. The polygon P is castable from a center of rotation r if the cast of P can be partitioned into exactly two parts, s.t. one part can be rotated in clockwise orientation around r and the other part can be rotated in counterclockwise orientation around r without intersecting the interior of P [3]. Hence, P is castable iff P is rotationally monotone. The algorithms presented in this paper are used by Bose et al. [2] to solve the casting problem in three dimensions. All proofs are available in the full version of the paper [3].

2 Preliminaries

Let P be a simple polygon in the plane with n vertices and let $int(P)$ and ∂P denote the interior and boundary of P, respectively, so that $P = int(P) \cup \partial P$. The edges of P are oriented in counterclockwise order s.t. $int(P)$ is located to their left. The aim is to determine whether

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the boundary of P can be partitioned into two pieces where each piece can be removed by a rotation.

For points r and p in the plane, we denote the circular arc with center r and angle α starting at p winding in clockwise (cw) or counterclockwise (ccw) direction by $cwarc(r, p, \alpha)$ or $ccwarc(r, p, \alpha)$ respectively.

Definition 1 *An edge e of P is* removable in cw orientation w.r.t. r *if* $\exists\ \alpha > 0$ *such that* $\forall\ p$ *on* e : $cwarc(r, p, \alpha) \cap int(P) = \emptyset$ *and* removable in ccw orientation w.r.t. r *if* $\exists\ \alpha > 0$ *such that* $\forall\ p$ *on* e : $ccwarc(r, p, \alpha) \cap int(P) = \emptyset$.

Then, we call the cw or ccw orientation a *valid removal orientation for e w.r.t. r* respectively, and we call r a valid center of rotation for e. Figure 1 illustrates the definition of removability for edges.

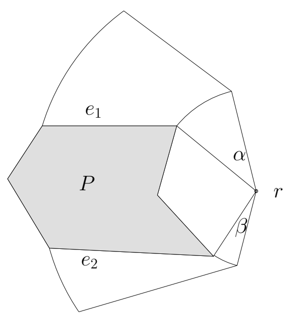

Figure 1: *The edges e_1 and e_2 are removable in cw orientation with angle α and ccw orientation with angle β w.r.t. r respectively.*

Definition 2 *Let r be a point in the plane. A polygon P is* rotationally monotone w.r.t. r, *if ∂P can be partitioned into exactly two connected chains, s.t. all edges of one chain are removable in cw orientation w.r.t. r and all edges of the other chain are removable in ccw orientation w.r.t. r.*

This implies that there exists an angle α, s.t. both chains can be rotated in cw or ccw orientation w.r.t. r, respectively, without colliding with each other. Note that the partitioning of the chain is not necessarily at vertices of P. We now outline a key property that characterizes all locations from which an edge is removable.

For an edge $e \in \partial P$ with incident vertices a and b, let $n_e(a)$ denote the line perpendicular to e passing through a. The line $n_e(a)$ divides the plane into two half planes and the open half plane containing b is denoted by $n_e^+(a)$ and the open half plane that does not contain b is denoted by $n_e^-(a)$. The supporting line $l(e)$ of e divides the plane into two half planes. Denote the open half plane located to the left of e when traversing P in ccw orientation by $l^+(e)$ and the open half plane located to the right of e when traversing P by $l^-(e)$, see Figure 2. The closure of an open set S is denoted by $cl(S)$.

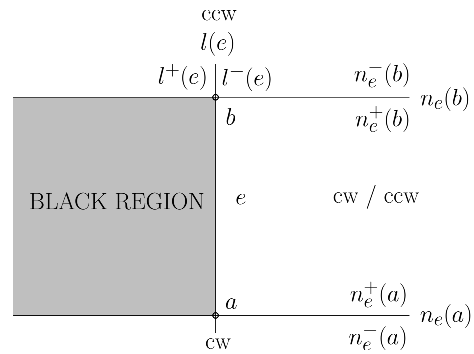

Figure 2: *The half planes associated with an edge e.*

Lemma 1 *Let e be an edge of P and denote the two vertices incident to e in ccw order by a and b.*

1. *The edge e is removable using a cw rotation around r, iff $r \in cl(n_e^-(a))$.*
2. *The edge e is removable using a ccw rotation around r, iff $r \in cl(n_e^-(b))$.*
3. *The point $r \in n_e^+(a) \cap n_e^+(b) \cap cl(l^-(e))$ iff the orthogonal projection of r on e partitions e into two parts, s.t. one part is removable using a cw rotation around r and the other part is removable using a ccw rotation around r.*
4. *The edge e is not removable, iff $r \in n_e^+(a) \cap n_e^+(b) \cap l^+(e)$.*

3 Decision Problem

In this section, we address the question of whether a polygon P is rotationally monotone w.r.t. a given point of rotation r and present an algorithm that solves the problem in linear time. If P is rotationally monotone w.r.t. r, the two points on ∂P, where the boundary of P is partitioned, need to be found.

Definition 3 *A point $c \in \partial P$ is said to be a* near point c w.r.t. r *if there exists a disk b centered at c with a positive radius, s.t. all points $q \in (\partial P \cap b) \setminus \{c\}$ are outside of the closed disk centered at r and passing through c.*

Hence, if c is not a vertex, c is the orthogonal projection of r on an edge of P. Therefore, c locally minimizes the distance between the boundary of P and the center of rotation r.

Definition 4 *A point $f \in \partial P$ is said to be a* far point f w.r.t. r *if there exists a disk b centered at f with a positive radius, s.t. all points $q \in \partial P \cap b$ are completely contained in the closed disk centered at r and passing through f.*

A far point is always a vertex of P that locally maximizes the distance between the boundary of P and the center of rotation r.

Definition 5 *Let $p \in \partial P$. If p is located in the interior of an edge, split the edge into two edges at p. The valid removal orientation w.r.t. r is said to* change *at p if one of the edges incident to p is removable in cw orientation and the other edge incident to p is removable in ccw orientation w.r.t. r.*

Lemma 2 *The valid removal orientation w.r.t. r changes at a point $p \in \partial P$ iff p is either a near point or a far point w.r.t. r.*

Theorem 3 *Given a center of rotation r, a polygon P is rotationally monotone w.r.t. r iff there exists exactly one near point c w.r.t. r and exactly one far point f w.r.t. r on ∂P.*

Theorem 3 allows us to determine whether a polygon is rotationally monotone given a center of rotation r by testing how many points $p \in \partial P$ are local extrema w.r.t. the distance between p and r. The polygon is rotationally monotone iff there is exactly one maximum and one minimum.

Theorem 4 *Given a polygon P with n vertices and a center of rotation r, we can test in $O(n)$ time whether P is rotationally monotone w.r.t. r.*

4 All valid regions of rotational monotonicity

In this section, the aim is to find all points r in the plane, s.t. a given polygon is rotationally monotone w.r.t. r.

Definition 6 *The union of all points r in the plane with the property that P is rotationally monotone w.r.t. r is the* valid region of rotational monotonicity of *P. The complement of the valid region is the* invalid region for rotational monotonicity of *P.*

The aim is to determine all valid regions in the plane for a given polygon P by partitioning the plane into valid and invalid regions for rotational monotonicity. Once a query point r is given, it is possible to determine whether r is a valid center of rotation for P by determining whether r is contained in a valid or an invalid region of rotational monotonicity.

4.1 Rotational monotonicity of convex polygons

In this section, we consider convex polygons and show that it is possible to find all valid regions of rotational monotonicity in linear time. The plane is partitioned into valid and invalid regions of rotational monotonicity by constructing the envelope of an arrangement of half lines.

Lemma 1 implies that every edge e with incident vertices a and b given in ccw order on ∂P splits the plane into regions of different valid removal orientations, see Figure 2.

Definition 7 *Let e be an edge of P and denote the two vertices incident to e in ccw order by a and b. The open strip $n_e^+(a) \cap n_e^+(b) \cap l^+(e)$ is the* black region *of e.*

Note that the black region does not contain any valid centers of rotation r for which e is removable (see Lemma 1, case 4).

Lemma 5 *A convex polygon P is rotationally monotone w.r.t. a point r iff r is not in the union of all black regions of edges of P.*

Lemma 6 *The valid region of rotational monotonicity of a convex polygon P consists only of unbounded regions in the plane.*

Based on Lemma 5 and Lemma 6, we compute the boundary of the union of all black regions of edges of P. For this, the notion of an envelope of n lines is defined.

Definition 8 *A set of n lines in the plane induces a subdivision S of the plane. The* envelope of the n lines *is the polygon formed by the bounded edges of all the unbounded regions of S [7].*

Similarly, a convex polygon P and the half lines bounding the black regions of its edges induce a subdivision S of the plane. Parallel half lines with the same orientation intersect at infinity and are therefore considered to be bounded edges. The polygon formed by the bounded edges of all the unbounded regions of S is called the envelope of the arrangement induced by *P.*

Lemma 6 implies that all valid regions of rotational monotonicity of P are contained in the complement of the envelope of the arrangement induced by P. This can be computed in linear time by modifying the algorithm by Keil [7] for computing envelopes of arrangements of lines.

Theorem 7 *Given an n-vertex convex polygon P, a description of the valid regions of rotational monotonicity of P has $O(n)$ size and can be computed in $O(n)$ time.*

Corollary 8 *A convex polygon P with n vertices can be preprocessed in $O(n)$ time, s.t. for any given point r, we can decide in $O(\log n)$ time if P is rotationally monotone w.r.t. r.*

4.2 Rotational monotonicity of simple polygons

In this section, we consider simple (not necessarily convex) polygons with n vertices and show that it is possible to find all valid regions of rotational monotonicity of P in $O(n^2)$ time. If the aim is to report all valid regions, this time bound is worst case optimal.

Let r be a point in the plane. If the valid removal orientation of a simple polygon P changes w.r.t. r at a reflex vertex $v \in \partial P$, v penetrates $int(P)$ when rotated infinitesimally around r with arbitrary orientation. This yields the following observation:

Observation 1 *A rotationally monotone polygon P w.r.t. r cannot be divided at one of its reflex vertices v unless the center of rotation r is v. Hence, v cannot be a far point w.r.t. r and v can only be a near point w.r.t. r if $r = v$.*

Definition 9 *Let v be a vertex of P and denote the two edges adjacent to v by e_1 and e_2. The* near cone *of v is defined as $cl(n^-_{e_1}(v) \cap n^-_{e_2}(v))$ and denoted by $NC(v)$.*

The near cone of v is the set of all points $X \in \mathbb{R}^2$ with the property that v is a near point w.r.t. X, see Figure 3.

Definition 10 *Let v be a vertex of P and denote the two edges adjacent to v by e_1 and e_2. The* far cone *of v is defined as $n^+_{e_1}(v) \cap n^+_{e_2}(v)$ and denoted by $FC(v)$.*

The far cone of v is the set of all points $X \in \mathbb{R}^2$ with the property that v is a far point w.r.t. X, see Figure 3.

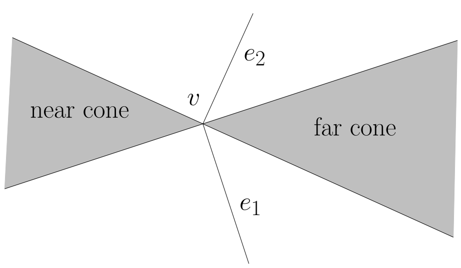

Figure 3: *The near cone and the far cone of v.*

Definition 11 *The* black region *of a reflex vertex v is $(NC(v) \cup FC(v)) \setminus \{v\}$.*

Note that Observation 1 ensures that the black region of v does not contain any valid centers of rotation r that allow the removal of v from ∂P.

Lemma 9 *A polygon P is rotationally monotone w.r.t. a point r iff r is not in the union of all black regions of edges and reflex vertices of P.*

Theorem 10 *Given a simple polygon P with n vertices, a description of the valid regions of rotational monotonicity of P has $O(n^2)$ size and can be computed in $O(n^2)$ time.*

Corollary 11 *A simple polygon P with n vertices can be preprocessed in $O(n^2)$ time, s.t. for any given point r, we can decide in $O(\log n)$ time if P is rotationally monotone w.r.t. r.*

In the full version [3], we construct a simple polygon having a quadratic number of disjoint valid regions. Therefore, Theorem 10 is worst-case optimal.

5 Future Work

The definition of rotational monotonicity w.r.t. a point r only tests whether the boundary of the polygon P can be decomposed into two chains, s.t. both chains can be rotated around r by a small angle without colliding with $int(P)$. An interesting extension is to determine whether the two chains can be rotated by a given angle α without colliding with the interior of the polygon. Another related problem is to find the maximal angle α the two chains can be rotated by without colliding with the interior of the polygon for a rotationally monotone polygon w.r.t. r.

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