Establishing Strong Connectivity using Optimal Radius Half-Disk Antennas

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Abstract

Given a set $S$ of points in the plane representing wireless devices, each point equipped with a directional antenna of radius $r$ and aperture angle $\alpha \geq 180^\circ$, our goal is to find orientations and a minimum $r$ for these antennas such that the induced communication graph is strongly connected. We show that $r = \sqrt{3}$ if $\alpha \in [180^\circ, 240^\circ)$, $r = \sqrt{2}$ if $\alpha \in (240^\circ, 270^\circ)$, $r = 2\sin(36^\circ)$ if $\alpha \in [270^\circ, 288^\circ)$, and $r = 1$ if $\alpha \geq 288^\circ$ suffices to establish strong connectivity, assuming that the longest edge in the Euclidean minimum spanning tree of $S$ is 1. These results are worst-case optimal and match the lower bounds presented in Caragiannis et al., “Communication in wireless networks with directional antennae”, SPAA’08. In contrast, $r = 2$ is sometimes necessary when $\alpha < 180^\circ$.

1 Introduction

Consider a wireless network modeled by a set of planar point sites $S$, each equipped with a transceiver having a transmission radius $r$. Typically one

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assumes that communication is omni-directional and two nodes can directly communicate with each other if the distance separating them is \( r \) or less. Geometrically, the transmission region of an antenna at a point \( p \) is modeled by a circle of radius \( r \) centered at \( p \). The connectivity of the network can be represented by a communication graph \( G(S) \), which has a node for each point and an edge between each pair of nodes separated by distance \( r \) or less.

Recently there has been interest in using directional antennas in place of their omni-directional counterparts [3, 5–8, 10, 11]. Some advantages of using directional antennas are enhanced security and reduced communication interference. Furthermore, if directional antennas are cleverly used, the power consumption of the network may be reduced. The transmission region of a directional antenna at a node \( p \) is geometrically represented by the sector of a circle with its apex at \( p \), a central angle \( \alpha \), and a radius \( r \). Its orientation is determined by a rotation \( \theta \) about \( p \). We assume that all antennas have the same \( \alpha \) and \( r \); it is only \( \theta \) that varies. Thus communication between two nodes is no longer symmetric and is best modeled by a directed communication graph in which a directed edge \( \overrightarrow{pq} \) indicates that \( q \) lies in \( p \)'s sector.

The direction assignment problem is the task of finding orientations for a set of directional antennas such that the induced communication graph has certain desired properties. In this paper we focus on obtaining a strongly connected communication graph using minimal \( r \). We will assume that \( S \) is normalized so that the length of the longest edge in its Euclidean minimum spanning tree is 1 (otherwise, we can scale the problem instance so that this property is satisfied). It is easy to see that to achieve connectivity in the normalized point set, \( r \) must be at least 1. Non-trivial lower bounds between the minimum communication radius and \( \alpha \) were given by Caragiannis et al. [5] (the exact bounds can be seen in Table 1). These bounds correspond to the case in which there is one node \( p \) located at the origin and \( k \) other nodes forming a regular \( k \)-gon inscribed in the unit circle centered at \( p \) (for \( k = 3, 4, 5 \) and 6).

In this paper, we show that the lower bounds of Caragiannis et al. are tight for any \( \alpha \geq 180^\circ \). Our proofs are constructive. That is, we design algorithms, for any \( \alpha \geq 180^\circ \), that orient the antennas at a given set of nodes so as to construct a strongly connected network in which no antenna is assigned a communication radius larger than the corresponding bound listed in Table 1.

In addition to providing lower bounds on \( r \), Caragiannis et al. [5] also gave an algorithm for orienting antennas with \( 180^\circ \leq \alpha < 288^\circ \) to obtain strong connectivity using \( r = 2 \sin(180^\circ - \alpha / 2) \) (notice that when \( \alpha \geq 288^\circ \) the problem is trivial). Thus, the algorithms presented here strictly improve their methods

<table>
<thead>
<tr>
<th>Aperture</th>
<th>(180°, 240°)</th>
<th>(240°, 270°)</th>
<th>(270°, 288°)</th>
<th>(288°, 360°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bound</td>
<td>( \sqrt{3} )</td>
<td>( \sqrt{2} )</td>
<td>( 2\sin36^\circ \approx 1.1756 )</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1: Communication radius needed to achieve connectivity as a function of the aperture angle \( \alpha \) (for \( \alpha \geq 180^\circ \)).
for the interval $\alpha \in [180^\circ, 288^\circ$). Damian and Flatland [8] consider directional antenna of fixed aperture angles of $120^\circ$ and $90^\circ$, and provide bounds of $r = 5$ and $r = 7$ (resp.) while at the same time bounding the number of hops to 5 and 6 (resp.) for nodes within unit distance. Bose et al. [11] have recently shown that a connected network using omni-directional communication can be replaced with directional antennas (with any $\alpha > 0^\circ$) so that the increase of $r$ and hop distance are bounded by constant factors (which depend on $\alpha$). In particular, they show that for $\alpha < 60^\circ$ and $k = 360^\circ/\alpha$, a radius of $4\sqrt{2}(3.5k - 6)$ suffices to establish a strongly connected communication network; and for $\alpha \geq 60^\circ$, a radius of $4\sqrt{2}(3 + k)$ suffices to establish a strongly connected communication network. (In both cases, the network turns out to be a hop spanner of the unit disk graph as well.)

Nijnatten [3] considered a variant of this problem, in which a different radius is allowed for each antenna, and the goal was to minimize the overall power consumption of the network. Ben-Moshe et al. [7] also considered similar problems for $90^\circ$-antennas but restricted the orientations of the nodes to one of the four standard quadrant directions. Bhattacharya et al. [6] considered nodes with multiple directional antennas and focused on minimizing the sum of the antenna angles for a fixed $r$. Kranakis et al. [10] have recently published a survey of results pertaining to the use of directional antennas in wireless networks.

**Results and Organization** The main result of this paper is the following.

**Theorem 1.** For any set $S$ of $n > 0$ points in the plane and $\alpha \geq 180^\circ$, there exists an orientation of antennas of angle $\alpha$ at the points of $S$ so that the communication graph is strongly connected, and the transmission radius of each antenna is at most $\sqrt{3}$ if $\alpha \in [180^\circ, 240^\circ)$, $\sqrt{2}$ if $\alpha \in [240^\circ, 270^\circ)$, $2\sin(36^\circ)$ if $\alpha \in [270^\circ, 288^\circ)$, and 1 if $\alpha \geq 288^\circ$. Moreover, such orientations can be found in $O(n)$ time, provided that the minimum spanning tree of $S$ is known in advance.

In Section 2, we present the basic algorithm for $\alpha \geq 180^\circ$ and $r = \sqrt{3}$. A proof of correctness for the algorithm is given in Section 3. Finally, in Section 4 we give the necessary modifications to reduce the communication radius to the lower bounds given in Table 1 for values of $\alpha$ in the ranges $[240^\circ, 270^\circ)$, $[270^\circ, 288^\circ)$, and $\geq 288^\circ$.

### 2 Orienting Antennas for $\alpha = 180^\circ$

In this section, we establish an upper bound by presenting an algorithm for orienting $180^\circ$-antennas of radius $r = \sqrt{3}$ to obtain a strongly connected communication graph. Let $\text{MST}_5$ be a minimum spanning tree of $S$ with maximum degree of five, such as the one described in [1]. Our algorithm processes nodes in the order in which they are visited in a breath-first traversal of $\text{MST}_5$. When a node is visited, it is assigned other nodes (within distance $\sqrt{3}$) for its antenna to cover (so as to satisfy certain invariants). If a node $v$ is assigned to cover node
w, we will say that “v points to w,” and we use the notation v → w. During
the traversal of MST₅, nodes are colored white, gray, or black. Initially all nodes
are white, meaning that they have not yet been visited and do not point to any
nodes. Visited nodes are black, and they point to at least one and at most two
other gray or black nodes. Gray nodes are direct children of visited nodes but
have not yet been visited themselves. They point to one other gray or black
node.

Let the gray/black communication subgraph be the graph consisting of the
gray/black nodes and having a directed edge u → w between each pair of nodes
(u, w) such that u → w. Our goal is to assign/adjust what the nodes point
to by inserting/updating edges of length at most \(\sqrt{3}\) in the gray/black com-
unication subgraph such that, throughout the tree traversal, the gray/black
communication subgraph is strongly-connected. For each gray/black node, ob-
serve that it is trivial to determine an orientation for its 180°-antenna that
covers the one or two nodes it points to. We note that the full communication
graph induced by these nodes may include additional edges, since a node’s 180°-
antenna may (by chance) cover nodes in addition to the one or two explicitly
assigned to it, but these edges are not needed for strong connectivity.

Let the root of MST₅ be any node with degree one. To get started, we color
the root node black and its child gray, and we constrain them to point to each
other. Starting with the root’s child, we visit the nodes one by one in a breadth-
first search order. When a node v is visited, it is initially gray. During the visit,
we change its color from gray to black and change the color of its children from
white to gray. We then locally update/insert directed edges in the gray/black
communication subgraph so that the following invariants are satisfied:

(I1) Each black node points to at least one and at most two gray/black nodes.

(I2) Each gray node points to exactly one gray/black node.

(I3) For each gray node v, one of the following is true:

(I3a) v points to its parent p. (Figure 1a)

(I3b) p points to v. (Figure 1b)

(I3c) p has children s and d that are (resp.) the first child clockwise and
first child counter-clockwise from v, and p → s → v → d → p. In
addition, \(\angle spv + \angle vpd \leq 180°\), and s and d lie on opposite sides of
line \(l_{vp}\) (Figure 1c).

(I4) The gray/black communication subgraph has edges no longer than \(\sqrt{3}\)
and is strongly connected.

We describe inductively on the number of black nodes our method for main-
taining these invariants. In the base case there is one black node, the root of
MST₅, and it points to its single gray child, which points back to the root. Ob-
serve that Invariant (I1) holds for the root and Invariants (I2, I3a) hold for its
child. Also, observe that Invariant (I4) holds for the root and its child. Assume
Figure 1: Solid edges are $\text{MST}_5$ edges; the arrows represent directed edges in the gray/black communication graph; the dotted arrow in (b,e) represents $v$’s directed edge to some other gray/black node (by Invariant (I2)). (a-c) Invariants (I3a), (I3b) and (I3c). (d-f) Edges in the gray/black communication graph are inserted/updated when $v$ is visited.

Let $v$ be the $(i+1)$-st node visited. We introduce some definitions so that we can process $v$ in a uniform manner, independent of its degree. Let $v_0, v_1, \ldots, v_{k-1}$ (for some $k \leq 5$), be the nodes adjacent to $v$ in counter-clockwise order, with $v_0 = p$. We define $\text{children}(v) = (v_1, \ldots, v_{k-1})$ as the ordered list containing all children of $v$, sorted angularly. We also define $\text{boundary}(v)$ as the set containing the extreme (first and last) elements of the ordered list $\text{children}(v)$. More formally, if $\deg(v) = 1$, then $\text{boundary}(v) = \emptyset$; otherwise, $\text{boundary}(v) = \{v_1, v_{k-1}\}$ (these nodes are referred to as boundary children). We say that the boundary child $v_1$ is isolated if $\deg(v) \geq 3$ and $\angle v_1vv_2 > 120^\circ$. Similarly, the boundary child $v_{k-1}$ is isolated if $\deg(v) \geq 3$ and $\angle v_{k-2}vv_{k-1} > 120^\circ$. Non-boundary children are never isolated.

In Figure 1b for example, $v_3$ is isolated, but neither $v_1$ nor $v_2$ is isolated. Let $\text{isolated}(v)$ be the set containing all isolated children of $v$. By our definition $\text{isolated}(v)$ contains zero, one or two boundary children of $v$, therefore it has cardinality at most 2.

We define the predicate $\text{in-range}(v, w) = \text{true}$ if and only if $\text{dist}(v, w) \leq \sqrt{3}$.

For any gray node $v$, let $\text{source}(v)$ be the node pointing to $v$, and let $\text{dest}(v)$ be
the node that $v$ points to (recall that a gray node always points to a single other node). For instance, if $v$ satisfies (I3a), then $\text{dest}(v) = p$; if $v$ satisfies (I3b), then $\text{source}(v) = p$; and if $v$ satisfies (I3c), then $\text{source}(v) = s$ and $\text{dest}(v) = d$ (see Figure 1c). By Invariant (I4) $\text{source}(v)$ and $\text{dest}(v)$ are well defined.

Algorithm 1 details how we update/insert edges in the gray/black communication subgraph when visiting $v$. We begin by describing the operation of the algorithm. The IF statement in lines 1-13 initializes the variables $v_{\text{from}}$ and $v_{\text{to}}$ to be two nodes with a directed edge between them such that one of the two nodes is $v$. The existence of such an edge is guaranteed by Invariant (I3). For example, in Fig 1b, $v_{\text{from}} = p$ and $v_{\text{to}} = v$. In Figure 1c, there are two directed edges incident to $v$, one of which will be used to initialize $v_{\text{from}}$ and $v_{\text{to}}$: in this example, there are no isolated children and $v_4$ is within range of $d$, so we set $v_{\text{from}} = v$ and $v_{\text{to}} = d$.

The remaining pseudocode (lines 14-20) first determines if there is a boundary child of $v$ that is within distance $\sqrt{3}$ of both $v_{\text{from}}$ and $v_{\text{to}}$. If so, then variable $v_{\text{via}}$ is initialized to one such child, with preference to an isolated boundary child (lines 16-17). For example, in Figure 1b, $v_{\text{via}} = v_3$; in Figure 1c, $v_{\text{via}} = v_4$. Then the algorithm does two things. First, it replaces the edge $v_{\text{from}} \rightarrow v_{\text{to}}$ with the two edges, $v_{\text{from}} \rightarrow v_{\text{via}}$ and $v_{\text{via}} \rightarrow v_{\text{to}}$ (line 19). This incorporates child $v_{\text{via}}$ into the strongly connected subgraph of gray/black nodes. Second, it calls the subroutine $\text{CHAIN}$ (line 20) which inserts edges that link $v$ and its children other than $v_{\text{via}}$ into a cycle. This incorporates the other children into the strongly connected subgraph of gray/black nodes. See figure pairs (1b, 1c) and (1c, 1f), showing before and after edge insertions. If, however, there is no boundary child in range of $v_{\text{from}}$ and $v_{\text{to}}$ in line 14, all of $v$’s children will be linked into a cycle through procedure $\text{CHAIN}$. In either case, all children of $v$ will be incorporated into the gray/black strongly connected subgraph.

We give some intuition regarding the isolated children and the algorithm’s preference for them. If $v$ has an isolated boundary child, $v'$, then we may not be able to $\text{CHAIN}$ $v'$ with the other children, because the angle between $v'$ and the next sibling (in clockwise or counter-clockwise order) is $> 120^\circ$, and thus the next sibling may be at a distance $> \sqrt{3}$. Observe that, in the portion of the IF statement involving Invariant (I3c) (lines 5-13), there is a preference for initializing $v_{\text{from}}$ and $v_{\text{to}}$ such that they both are in range of an isolated child. (Since one of these two variables will be set to $v$, which is within range of all its children, we only need to check if $\text{dest}(v) = d$ or $\text{source}(v) = s$ is within range.) This is done so that $v_{\text{via}}$ will be set to an isolated child (in line 18), and thus an isolated child undergoes the $\text{REPLACE}$ operation rather than being chained with the other children. (In Section 3 we prove that the remaining children can be chained.) If no isolated child is within range in lines 6-9, then the algorithm attempts to set $v_{\text{from}}$ and $v_{\text{to}}$ so that they are within range of a regular boundary child (lines 10-13). The reason for this is that in order to maintain our invariants, we must not chain more than 3 children. Thus, if there is a boundary child in range, then it undergoes the $\text{REPLACE}$ operation, whereas the other (at most 3) children are chained.
Algorithm 1: Visit(Node v)

1) if \( v \rightarrow p \) then /* Invariant I3a */
2) \( \text{v}_\text{from} = v, \text{and} \text{v}_\text{to} = p \)
3) else if \( p \rightarrow v \) then /* Invariant I3b */
4) \( \text{v}_\text{from} = p, \text{and} \text{v}_\text{to} = v \)
5) else /* Invariant I3c */
6) if \( \exists v' \in \text{isolated}(v) \text{ s.t. in-range}(v', \text{dest}(v)) \) then
7) \( \text{v}_\text{from} = v, \text{v}_\text{to} = \text{dest}(v) \)
8) else if \( \exists v' \in \text{isolated}(v) \text{ s.t. in-range}(v', \text{source}(v)) \) then
9) \( \text{v}_\text{from} = \text{source}(v), \text{v}_\text{to} = v \)
10) else if \( \exists v' \in \text{boundary}(v) \text{ s.t. in-range}(v', \text{dest}(v)) \) then
11) \( \text{v}_\text{from} = v, \text{v}_\text{to} = \text{dest}(v) \)
12) else
13) \( \text{v}_\text{from} = \text{source}(v), \text{v}_\text{to} = v \)
14) if \( \exists v' \in \text{boundary}(v) \text{ s.t. in-range}(v', \text{v}_\text{from}) \land \text{in-range}(v', \text{v}_\text{to}) \) then
15) \( \text{v}_\text{via} = v' \)
16) if \( \exists v' \in \text{isolated}(v) \text{ s.t. in-range}(v', \text{v}_\text{from}) \land \text{in-range}(v', \text{v}_\text{to}) \) then
17) \( \text{v}_\text{via} = v' \)
18) REMOVE \( \text{v}_\text{via} \) from children(\( v \))
19) REPLACE \( \text{v}_\text{from} \rightarrow \text{v}_\text{to} \) with \( \text{v}_\text{from} \rightarrow \text{v}_\text{via} \) and \( \text{v}_\text{via} \rightarrow \text{v}_\text{to} \)
20) CHAIN(\( v, \text{children}(v) \))

Subroutine 2: CHAIN(\( v, (v'_1, v'_2, \ldots, v'_\ell) \))

Add edges: \( v \rightarrow v'_1 \rightarrow v'_2 \rightarrow \ldots v'_{\ell-1} \rightarrow v'_\ell \rightarrow v \)

3 Proof of Correctness

Before proving Theorem 1 for \( \alpha \in [180^\circ, 240^\circ) \), we first introduce some helpful lemmas. Let \( \ell_{pq} \) be the line passing through points \( p \) and \( q \) (the segment connecting them will be denoted by \( pq \)).

Lemma 1. Let \( (a, b, c, d) \) be a path in a minimum spanning tree \( T \) such that \( a \) and \( d \) lie on or to a same side of \( \ell_{bc} \). Then \( \angle abc + \angle cda > 150^\circ \).

Lemma 2. Let \( (a, b, c, d) \) be a path in MST$_5$ such that \( a \) and \( d \) lie on or to the same side of the line \( \ell_{bc} \). Furthermore, \( 60^\circ \leq \angle abc \leq 150^\circ \), \( 60^\circ \leq \angle cda \leq 150^\circ \), and \( \angle abc + \angle cda \leq 210^\circ \). Then \( |ad| \leq \sqrt{3} \).

These two results hold for the Euclidean minimum spanning tree of any set of points. Since the proofs of the above claims are fairly long and offer no additional insight into the workings of our algorithm, we defer their proofs to the appendix.
Lemma 3. If $\deg(v) \geq 4$, then $v_{via}$ is initialized. Furthermore, if $v$ has an isolated child, then $v_{via}$ is initialized to an isolated child.

Proof. We show that the condition of the IF statement on line 14 of the algorithm evaluates to true, so $v_{via}$ gets initialized on line 15. If $v$ satisfies Invariant (I3a) or (I3b), then $v_{to}, v_{from} \in \{v, p\}$. Note that at most one boundary child $v'$ of $v$ may satisfy $\angle v'vp > 120^\circ$, because each angle between radially consecutive children of $v$ is at least $60^\circ$, and the sum of all these angles is $360^\circ$. It follows that the second boundary child (which always exists, because $\deg(v) \geq 4$) is within range of both $p$ and $v$, therefore the condition of the IF statement on line 14 evaluates to true and $v_{via}$ is initialized on line 15. By similar arguments, if $v$ has an isolated boundary child, say $v_1$, then with the exception of $\angle v_1v_2$, all other angles at $v$ must be smaller than $120^\circ$. Thus $v_1$ is within range of $p$ and $v$, and therefore $v_{via}$ is initialized to an isolated child in line 17.

Next we discuss the more complex situation when $v$ satisfies Invariant (I3c), so $v$ is involved in a cycle $p \rightarrow s \rightarrow v \rightarrow d \rightarrow p$. (See for example Figure 1c.)

First recall that by Invariant (I3c), $s$ and $d$ lie to opposite sides of the line $\ell_{vp}$, and $s$, $v$ and $d$ are radially consecutive children of $p$, in counter-clockwise order. Since $\deg(v) \geq 4$ and radially consecutive adjacent edges in an MST form an angle of at least $60^\circ$, boundary children $v_1$ and $v_{k-1}$ (where $k = \deg(v)$) cannot lie on the same side of the line $\ell_{vp}$. Also recall that $v_1$, $p$ and $v_{k-1}$ are radially consecutive neighbors of $v$, in clockwise order. It follows that $s$ and $v_1$ are both on or to one side of the line $\ell_{vp}$, and $d$ and $v_{k-1}$ are both on or to the other side. We will use this fact when applying Lemma 2 below.

Consider first the case when $\deg(v) = 4$. We discuss two situations, depending on whether $v$ has isolated children or not. Assume first that $v$ has no isolated children, as in Figure 2a. Note that $\angle v_1vp + \angle v_3vp \leq 240^\circ$, because each of $\angle v_1v_2$ and $\angle v_2v_3$ is at least $60^\circ$, and the sum of all these angles is $360^\circ$. These together imply that $\angle v_1vp + \angle v_3vp + \angle spd \leq 240^\circ + 180^\circ = 420^\circ$, so at least one of $\angle v_3vp + \angle vp d$ and $\angle v_1vp + \angle vps$ is no greater than $210^\circ$. For the pair

![Figure 2: Lemma 3: Case of Invariant (I3c), $\deg(v) = 4$, and (a) no isolated children (b) an isolated child.](image-url)
whose angle sum is no more than 210°, each individual angle is at least 60° and
no more than 210° - 60° = 150°. Having verified the requirements of Lemma 2
for one of the two paths, \((v_1, v, p, s)\) or \((v_3, v, p, d)\), we use it to show that either
\(\text{in-range}(v_3, d) = \text{true}\) or \(\text{in-range}(v_1, s) = \text{true}\) (or both). If \(\text{in-range}(v_3, d) = \text{true}\),
then \(v_{\text{from}}\) and \(v_{\text{to}}\) are initialized in line 11 of the algorithm; otherwise, \(v_{\text{from}}\) and
\(v_{\text{to}}\) are initialized in line 13 of the algorithm. In either case, the condition of the
IF statement in line 14 of the algorithm evaluates to \(\text{true}\).

Assume now that \(v\) has an isolated child, say \(v_1\), as in Figure 2b. By def-
inition, \(\angle v_1v_2v_3 > 120°\). This along with the fact that \(\angle v_2v_3v_1 \geq 120°\) implies
that \(\angle v_1v_2 + \angle v_3v_1 \leq 180°\). It follows that \(\angle v_1v_2 + \angle v_3v_1 + \angle vpd \leq 360°\). By
Lemma 1, \(\angle v_1v_2 + \angle vpd > 150°\). These together imply that \(\angle v_1v_2 + \angle vps \leq
210°\), and each of these angles has a value in the interval \([60°, 150°]\). By
Lemma 2, \(\text{in-range}(v_1, s) = \text{true}\). Then \(v_{\text{from}}\) and \(v_{\text{to}}\) are initialized in line 9
of the algorithm, and the conditions of both IF statements in lines 14 and 16 of
the algorithm evaluate to \(\text{true}\).

Consider now the case when \(\deg(v) = 5\). In this case, \(v\) has no isolated
children: each angle at \(v\) is at least 60°, the sum of all five angles is 360°,
therefore each angle is at most 120°. It follows that \(\angle v_1v_2v_3 \leq 240°\).
(In fact, a stronger upper bound is 180°, but this is irrelevant to the discussion
here.) This situation is identical to the degree 4, no isolated children case. □

**Lemma 4.** If \(\deg(v) \geq 3\), let \(v'_1, \ldots, v'_\ell \in \text{children}(v) \setminus \{v_{\text{via}}\}\) be radially sorted
around \(v\). Then, \(\text{in-range}(v'_i, v'_{i+1}) = \text{true}\) for \(i = 1, \ldots, \ell - 1\).

**Proof.** Recall that when \(\deg(v) = 5\), no angle between two radially consecutive
children of \(v\) exceeds 120°, and so the lemma is clearly true. So consider the sit-
tuation where \(\deg(v) < 5\). By similar arguments, at most one angle between two
radially consecutive children of \(v\) may exceed 120°. Furthermore, one of these
children is necessarily a boundary (isolated) child, because all angles between
radially consecutive children involve a boundary child when \(v\) is of degree 3 or 4.
As noted previously, a degree 4 vertex can have at most one angle > 120°. So if
\(v\) is of degree 4 and has an isolated child, then both its boundary children form
an angle < 120° with \(p\), and thus both are within range of \(p\). When \(\deg(v) = 3,\)
if one child is isolated, then they both are (since there are only two children.)
In this case, at least one of the two children must be within range of \(p\) or else
the sum of the three angles at \(v\) is more than 360°. If \(v\) satisfies Invariant \((I3a)\)
or \((I3b)\), then \(v_{\text{via}}, v_{\text{from}} \in \{v, p\}\), therefore the conditions of both IF statements
on lines 14 and 16 evaluate to \(\text{true}\). It follows that \(v_{\text{via}}\) is set to an isolated child
of \(v\) in line 17, and \(\text{children}(v) \setminus \{v_{\text{via}}\}\) contains either one child of \(v\) (the degree
3 case), or two children of \(v\) within range of each other (the degree 4 case).

It remains to discuss the more complex situation when \(v\) satisfies Invariant
\((I3c)\), so \(v\) is involved in a cycle \(v \rightarrow d \rightarrow p \rightarrow s \rightarrow v\), and \(\angle vpd + \angle vps \leq 180°\).
Assume without loss of generality that \(v_1\) is isolated, and \(v_1\) and \(s\) lie on the
same side of \(vp\) (refer to Figure 1c). If \(\deg(v) = 3\), then \(\angle v_1v_2 + \angle v_2vp \leq
240°\) (because \(v_1\) and \(v_2\) are both isolated, by our assumption). Arguments
similar to the ones used in the proof of Lemma 3 show that in this case either
in-range(v₂, d) = true, or in-range(v₁, s) = true, or both. If in-range(v₂, d) = true, 
v₁ and v₂ are initialized in line 7 of the algorithm; otherwise, v₁ and 
v₂ are initialized in line 9 of the algorithm. In either case, the conditions of 
both IF statements in lines 14 and 16 of the algorithm evaluate to true, and 

children(v) \ {v₁}\ contains a single child of v. 

If deg(v) = 4, Lemma 3 shows that in-range(v₁, s) = true. This guarantees 
that line 11 of the algorithm gets executed and v₁ gets removed from the 
list of children to be chained. It follows that children(v) \ {v₁}\ contains two 
children of v within range of each other. 

We now prove that Algorithm 1 is correct. We begin by proving that the 
CHAIN and REPLACE operations only add edges between nodes that are in 
range of each other and that they maintain Invariants (I1), (I2), and (I3). We 
then show that these operations also ensure that Invariant (I4) is satisfied. In 
what follows, let p(w) denote the parent of node w. 

Consider the REPLACE operation in line 19. Observe first that execution 
only reaches line 19 if in-range(v₁) = true, and therefore the edge updates are valid. We now verify that v₁ satisfies the invariants af- 
after REPLACE. If v satisfies (I3a), or if v satisfies (I3c) and in-range(v₁, dest(v)) = 
true, then v₁ = v. After REPLACE, v₁ will satisfy (I3b), since v₁ = v₁ → v₁ and v = p(v₁). Otherwise, v satisfies (I3b) and 
in-range(v₁, source(v)) = true, and so v₁ = v. After REPLACE, v₁ will satisfy 
(I3a), since v₁ → v₁ = v and v = p(v₁). It is easy to verify that v₁ satisfies 
(I2), and since REPLACE does not change the number of nodes pointed to by 
v₁, they continue to satisfy either (I1) or (I2). 

We now prove the correctness of the CHAIN operation, using four cases de- 
pending on the degree of node v. In each case, it is easy to verify that the 
children involved in CHAIN satisfy (I2) afterwards, since their color changes 
from white to gray and CHAIN makes them each point to one node. In addi-
tion, v satisfies (I1), since v points to one gray/black node before CHAIN, and 
CHAIN makes it point to one more gray/black node. Therefore, we focus on 
verifying (I3). In each case that follows, when v₁ = ∅, we assume v₁ is inti-
ialized to boundary child v₁ = v₁ → 1; situations in which v₁ is initialized to v₁ are 
analogous. 

• Case 1: (deg(v) = 1) ∨ (deg(v) = 2 ∧ v₁ = ∅). Note that in this case, 
there are no points in children(v) \ {v₁}. 

• Case 2: (deg(v) = 2 ∧ v₁ = ∅) ∨ (deg(v) = 3 ∧ v₁ = ∅). Then v₁ is 
the only child in children(v) \ {v₁}, and CHAIN adds edges v → v₁ → v. 
Since v = p(v₁), in-range(v₁, v₁) = true. Thus, v₁ satisfies (I3a). 

• Case 3: (deg(v) = 3 ∧ v₁ = ∅) ∨ (deg(v) = 4 ∧ v₁ = ∅). In this case, v₁ 
and v₂ are the two children in children(v) \ {v₁}, and CHAIN adds edges 
v → v₁ → v₂ → v. Note that, since v₁ = p(v₁) = p(v₁), in-range(v₁, v₁) = 
in-range(v₁, v₂) = true. Also, by Lemma 4, in-range(v₁, v₂) = true. Thus, 
v₁ satisfies (I3b) and v₂ satisfies (I3a).
• Case 4: \((\text{deg}(v) = 4 \land v_{\text{via}} = \emptyset) \lor (\text{deg}(v) = 5)\). Note that, if \(\text{deg}(v) \geq 4\), then \(v_{\text{via}} \neq \emptyset\) by Lemma 3. Therefore, we only need to handle the case when \(\text{deg}(v) = 5 \land v_{\text{via}} \neq \emptyset\). In this case, \(v_1\), \(v_2\), and \(v_3\) are the three children in \(\text{children}(v) \setminus \{v_{\text{via}}\}\), and CHAIN adds edges \(v \rightarrow v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v\). Note that, since \(v = p(v_1) = p(v_3)\), in-range\((v, v_1) = \text{in-range}(v, v_3) = \text{true}\). Also, in-range\((v_1, v_2) = \text{in-range}(v_2, v_3) = \text{true}\), by Lemma 4. Thus, \(v_1\) satisfies (I3b) and \(v_3\) satisfies (I3a).

To complete the proof, we show that \(v_2\) satisfies Invariant (I3c). First we verify that \(\angle v_1v_2v + \angle v_2v_3v \leq 180^\circ\). This is true because \(v_{\text{via}}\) is a boundary child of \(v\), and thus the remaining children \(v_1\), \(v_2\) and \(v_3\) are radially consecutive about \(v\). For a degree 5 node, any three radially consecutive adjacent nodes can span at most \(180^\circ\), since otherwise the sum of all five angles would be more than \(360^\circ\) (because the angle between radially consecutive adjacent edges in a MST\(_5\) is at least \(60^\circ\)). Finally, we verify that \(v_1\) and \(v_3\) are on opposite sides of the line \(\ell_{v_2v}\). For contradiction, suppose they are on or to the same side of this line. Because \(v_1, v_2, v_3\) are radially consecutive, this implies that all five nodes adjacent to \(v\) are on or to the same side of \(\ell_{v_2v}\), which again is impossible in a MST\(_5\).

We end by proving that Invariant (I4) is satisfied after visiting \(v\). Let \(G\) be the gray/black communication subgraph just prior to \(v\) being visited. By the inductive hypothesis, \(G\) is strongly connected. Consider the REPLACE operation. Observe that both \(v_{\text{from}}\) and \(v_{\text{to}}\) are gray/black nodes and therefore they are in \(G\). In addition, \(v_{\text{from}} \rightarrow v_{\text{to}}\) corresponds to an edge in \(G\). It is straightforward then to verify that adding node \(v_{\text{via}}\) to \(G\) and replacing edge \(v_{\text{from}} \rightarrow v_{\text{to}}\) in \(G\) with \(v_{\text{from}} \rightarrow v_{\text{via}}\) and \(v_{\text{via}} \rightarrow v_{\text{to}}\), results in a strongly connected graph. Similarly, adding to \(G\) \(v\)'s children, \(v_1', \ldots, v_5'\), involved in the CHAIN operation, along with the edges \(v \rightarrow v_1' \rightarrow \cdots \rightarrow v_5' \rightarrow v\), results in a strongly connected graph. Moreover, since \(v\)'s children are all colored gray when \(v\) is visited, Invariant (I4) is satisfied.

Finally, notice that each node is visited exactly once, and that a constant number of computations are needed at each node. It follows that the algorithm takes linear time, provided that MST\(_5\) is known in advance.

4 Larger Angular Values

For \(\alpha \in [180^\circ, 240^\circ]\), the algorithm presented is optimal in the sense that the antenna radius \(r = \sqrt{3}\) matches the known lower bound shown in Table 1; however, it does not match the known lower bounds for \(\alpha \geq 240^\circ\). In this section we show that, with minor modifications to the basic algorithm, for \(\alpha\) in each of the other three ranges in Table 1, an \(r\) equal to the corresponding lower bound is sufficient to establish a strongly connected communication graph.

In order to reduce the number of cases, we define constants \(c_\alpha\) that depend on the aperture angle \(\alpha\) as follows: \(c_{180^\circ} = 2\), \(c_{240^\circ} = 3\), \(c_{270^\circ} = 4\), and \(c_{285^\circ} = 5\). It is not difficult to verify that, by assuming an appropriate orientation, any
antenna of aperture $\alpha$ can cover $c_\alpha$ sensors, provided that they are within range.

We also define constants $r_\alpha$ corresponding to the lower bounds in Table 1 as follows: $r_{180^\circ} = \sqrt{3}$, $r_{240^\circ} = \sqrt{2}$, $r_{270^\circ} = 2\sin 36^\circ$, and $r_{288^\circ} = 1$. We define the function $\text{in-range}_\alpha(v_1, v_2)$ to be true if and only if $\text{dist}(v_1, v_2) \leq r_\alpha$. It is not difficult to verify that, if two nodes $v_1, v_2$ are both adjacent to a node $v$ in $\text{MST}_5$ and $\angle v_1v_2v \leq \frac{360^\circ}{\alpha + 1}$, then $\text{in-range}_\alpha(v_1, v_2) =$ true, since edge lengths in $\text{MST}_5$ are at most 1. (If $\angle v_1v_2v > \frac{360^\circ}{\alpha + 1}$, then $\text{in-range}_\alpha(v_1, v_2)$ may or may not be true.)

For antennas with $\alpha \geq 288^\circ$, obtaining a strongly connected communication graph with $r = 1$ is trivial, as observed in [5]: such antennas can cover any $c_{288^\circ} = 5$ other nodes within distance 1, and thus one can orient the antennas in a way that the obtained communication graph contains $\text{MST}_5$ as a subgraph.

It remains to consider the cases in which $\alpha \in [240^\circ, 270^\circ)$ and $\alpha \in [270^\circ, 288^\circ)$. For this purpose, we slightly redefine the concept of an isolated vertex. Let $v$ be a vertex of degree $k > c_\alpha$, $p$ its parent, and let $v_1, \ldots, v_{k-1}$ be its children and $v_0 = v_k = p$. We say that $v_1$ is isolated if $\angle pv_1 > \frac{360^\circ}{\alpha + 1}$, and analogously $v_{k-1}$ is isolated if $\angle v_{k-1}vp > \frac{360^\circ}{\alpha + 1}$.

**Lemma 5.** For $\alpha \in \{240^\circ, 270^\circ\}$, let $v$ be any vertex in $\text{MST}_5$ such that $\deg(v) > c_\alpha$. Vertex $v$ has either (a) an non-isolated child $c$ such that $\text{in-range}_\alpha(c, p) =$ true, or (b) two consecutive children $v_i, v_{i+1}$ such that $\text{in-range}_\alpha(v_i, v_{i+1}) =$ true. Moreover, if $\alpha = 240^\circ$ and $\deg(v) = 5$, there exists at most one index $0 \leq i < \deg(v)$ such that $\text{in-range}_{240^\circ}(v_i, v_{i+1}) =$ false.

**Proof.** Recall that, for any $0 \leq i < \deg(v)$, if $\angle v_i v_{i+1}v \leq \frac{360^\circ}{\alpha + 1}$, we must have $\text{in-range}_\alpha(v_i, v_{i+1}) =$ true. By the pigeonhole principle there must exist an index $i$ such that $\angle v_i v_{i+1}v \leq \frac{360^\circ}{\alpha + 1}$ (since otherwise the sum of angles $\angle v_i v_{i+1}v$ exceeds $360^\circ$). In particular, we either obtain a non-isolated child (if $i = 0$ or $i = \deg(v) - 1$) or we obtain two consecutive children that are in range.

For the second claim, if $\text{in-range}_{240^\circ}(v_i, v_{i+1}) =$ false, then $\angle v_i v_{i+1}v > \frac{360^\circ}{\alpha + 1} = 90^\circ$. Recall that any pair of radially consecutive adjacent edges in a minimum spanning tree form an angle of at least $60^\circ$. Together these imply that at a degree 5 vertex there is at most one index $i$ is such that $\text{in-range}_{240^\circ}(v_i, v_{i+1}) =$ false, since otherwise the angles at $v$ sum to more than $2(90^\circ) + 3(60^\circ) = 360^\circ$, a contradiction. \hfill \Box

With this result we can simplify the previous algorithm when $\alpha = 240^\circ$ and $\alpha = 270^\circ$. Intuitively, the idea is to add to the gray/black communication subgraph the edges $v \rightarrow v_i \rightarrow v$ from $v$ to all of its children $v_i$. When that is not possible (because it would require that $v$ cover more than $c_\alpha$ other nodes), we will reduce the number of nodes $v$ must cover by using Lemma 5. The invariants from Section 2 can be simplified to the following:

(11) Each black node points to at least one and at most $c_\alpha$ gray/black nodes.
(12) Each gray node points to exactly one gray/black node.
(13) Each gray node $v$ points to its parent $p$ (case I3a) or $p$ points to $v$ (case I3b).
The gray/black communication subgraph has edges no longer than \( r_\alpha \) and is strongly connected.

As before, the proof considers different cases according to the degree of \( v \). If \( \deg(v) \leq c_\alpha \), we simply add the edges \( v \rightarrow v_i \rightarrow v \) (for \( 1 \leq i < \deg(v) \)) to the gray/black communication graph. Notice that we add at most \( c_\alpha - 1 \) outgoing edges to \( v \), hence Invariant (I1) is satisfied. It is easy to see that other invariants also hold.

If \( \deg(v) = c_\alpha + 1 \), we will use Lemma 5 to save one outgoing edge at \( v \). Consider first the case (a) in which there exists a non-isolated child \( v_{\text{via}} \). In this case, we replace the edge \( v \rightarrow p \) or \( p \rightarrow v \) (one of those two edges must be present by Invariant (I3)) with \( v \rightarrow v_{\text{via}} \rightarrow p \) or \( p \rightarrow v_{\text{via}} \rightarrow v \), respectively. Now consider the case (b) when there exists an index \( 1 \leq j < \deg(v) - 1 \) such that in-range\((v_j, v_{j+1})\). We pick any index \( j \) that satisfies in-range\((v_j, v_{j+1})\) and add the cycle \( v \rightarrow v_j \rightarrow v_{j+1} \rightarrow v \). For all other indices \( i \neq j, j + 1 \) we add the edges \( v \rightarrow v_i \rightarrow v \). In both cases, exactly \( c_\alpha - 1 \) new outgoing edges are added to \( v \), and one edge is added to each of the children. All but exactly one child (either \( v_{\text{via}} \) or \( v_{j+1} \)) will satisfy Invariant (I3b), and this child will satisfy Invariant (I3a).

It remains to consider the case in which \( \deg(v) = c_\alpha + 2 \). Observe that, since the maximum degree of \( \text{MST}_5 \) is five, this can only happen when \( \alpha = 240^\circ \) and \( \deg(v) = 5 \), and also when \( \alpha = 270^\circ \) since \( c_{270^\circ} = 4 \) and there are no nodes \( v \) in \( \text{MST}_5 \) with \( \deg(v) = 6 \). For this case, we select a non-isolated vertex \( v_{\text{via}} \) and an index \( j \) such that \( v_j, v_{j+1} \neq v_{\text{via}} \). By the second claim of Lemma 5, this index will always exist. We now apply both of the above constructions to save two outgoing edges at \( v \). Specifically, we add edges \( v \rightarrow v_j \rightarrow v_{j+1} \rightarrow v \), and either \( v \rightarrow v_{\text{via}} \rightarrow p \) or \( p \rightarrow v_{\text{via}} \rightarrow v \). By doing so, we incorporate three children of \( v \) into the gray/black communication subgraph while increasing the number of outgoing edges at \( v \) by only 1. Finally, we connect the remaining child \( v_i \) of \( v \) by adding the edges \( v \rightarrow v_i \rightarrow v \). It is easy to see that all invariants are preserved, hence the proof of Theorem 1 is complete.

5 Closing Remarks

In this paper we show how to orient antennas with apertures no smaller than \( 180^\circ \) to form strongly connected networks. The antenna radii used in our approach are optimal in the sense that they match the lower bounds established by Caragiannis et al. [5] (\( \sqrt{3} \) for \( \alpha \in [180^\circ, 240^\circ) \), \( \sqrt{2} \) for \( \alpha \in [240^\circ, 270^\circ) \), \( 2 \sin 36^\circ \) for \( \alpha \in [270^\circ, 288^\circ) \)) and 1 for \( \alpha > 288^\circ \). A natural next step would be to extend our results to tighten the gap between the lower bounds and the existing upper bounds for \( \alpha < 180^\circ \). In terms of lower bounds, Caragiannis et al. [5] gave two simple problem instances that show \( r \geq \sqrt{7} \) for \( \alpha < 60^\circ \) and \( r \geq 2 \) for \( \alpha < 180^\circ \). For upper bounds, previous work established \( r = 60\sqrt{2} \) for \( \alpha < 60^\circ \) (more precisely, \( r = 4\sqrt{2}(3.5k - 6) \) where \( k = 360^\circ/\alpha \)) and \( r = 36\sqrt{2} \) for \( \alpha \geq 60^\circ \) (more precisely, \( r = 4\sqrt{2}(3 + k) \) where \( k = 360^\circ/\alpha \)) [11]. We note that the goal in [11] was to establish a hop spanner of the unit disk graph, in
addition to the strong connectivity property of the communication network. If however the hop spanning requirement is lifted, we believe that upper bounds matching the lower bounds from [5] could be established. For example, for any alpha value, an upper bound of 3 follows immediately from the fact that the cube \( \text{MST}_3 \) of \( \text{MST}_5 \) is Hamiltonian [2]: simply orient the antennas along the edges of \( \text{MST}_3 \), which are no longer than 3.

Distributed solutions to the direction assignment problem are also preferable due to the decentralized nature of wireless networks. Our approach cannot be easily extended to work in a distributed fashion, because it traverses the minimum spanning tree. Developing distributed approximation solutions to the direction assignment problem based on localized [9] or distributed [4] constructions of minimum spanning trees remains an interesting open problem.

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**References**


A Proof of Lemma 1

Lemma 1. Let \((a,b,c,d)\) be a path in a minimum spanning tree \(T\) such that \(a\) and \(d\) lie on or to a same side of \(\ell_{bc}\). Then \(\angle abc + \angle bed > 150^\circ\).

Proof. Let \(D(p,r)\) denote the open disk of radius \(r\) centered at point \(p\), and \(\partial D(p,r)\) denote its boundary. Rotate \(T\) so that \(bc\) is vertical, and \(a\) and \(d\) lie in the closed halfplane to the right of \(\ell_{bc}\). For simplicity, let \(|bc| = x\). Note that \(a\) must lie on or above the perpendicular bisector \(H\) of \(bc\). Otherwise, \(|ac| < |ab|\), so replacing \(ab\) by \(ac\) would result in a tree of smaller weight, contradicting the fact that \(T\) is a minimum spanning tree. Similarly, \(d\) must lie on or below \(H\). Also note that \(a\) may not lie in \(\text{lune}(b,c) = D(b,x) \cap D(c,x)\), because otherwise \(|ca| < |cb|\), so replacing \(bc\) by \(ac\) would result in a tree of smaller weight, a contradiction. The argument for \(d\) is symmetric. We label these properties, so that we can refer to them later:

- (P1) \(a\) lies on or above \(H\)
- (P2) \(d\) lies on or below \(H\)
- (P3) \(a\) and \(d\) lie outside \(D(b,x) \cap D(c,x)\)

The proof is by contradiction. Assume to the contrary that

\[ \angle abc + \angle bed \leq 150^\circ. \] (1)

This along with the fact that \(\angle abc \geq 60^\circ\) (a property of any Euclidean minimum spanning tree) implies that \(\angle bed \leq 90^\circ\). Symmetric arguments yield \(\angle abc \leq 90^\circ\).

Let \(r\) be a point on \(H\) such that \(\angle rbc = \angle rcb = 75^\circ\). Let \(p\) be the intersection point between \(\partial D(c,x)\) and \(rb\), and let \(q\) be the intersection point between \(\partial D(b,x)\) and \(rc\). See Figure 3.
We now argue that \(a\) and \(d\) cannot both lie in the closed triangle \(\triangle rbc\).

Let \(R = \triangle rbc \setminus \text{lune}(b,c)\) be a closed region. Let \(R^+\) be the closed half of \(R\) above \(\mathcal{H}\) and \(R^-\) be the closed half below (thus \(R^+\) and \(R^-\) share the points on \(R \cap \mathcal{H}\)). Assume for the sake of contradiction that both \(a,d \in \triangle rbc\). By properties (P1-P3), we have that \(a \in R^+\) and \(b \in R^-\). Assume \(a\) is on or to the right of a vertical line passing through \(d\). (Symmetric arguments hold if \(d\) is to the right of \(a\).) Then the point in \(R^-\) farthest to the left from \(a\) is \(q\), regardless of where \(a\) is in \(R^+\). Let \(k\) be the point at the intersection of \(\partial D(q,x)\) and the line segment \(rb\), as shown in Fig 3b. Note that \(\triangle cbq\) is isosceles, \(\angle cbq = 30^\circ\), \(\angle qbk = 75^\circ - 30^\circ = 45^\circ\), and therefore \(\angle qbk\) is a right angle.

Suppose \(a\) is in the region \(L\) defined to be the intersection of \(R^+\) and \(D(q,x)\).

(See shaded region in Figure 3b.) Since \(d = q\) is farthest from every point in \(L\), \(|ad| \leq |aq| < x\) (recall that \(D(q,x)\) is open, hence the strict inequality). Then \(|ad| < |bc|\), which contradicts the fact that \(T\) is a minimum spanning tree. Now suppose \(a\) is in the region \(R^+ \setminus L\). Then \(|ab| \geq |bk| = x\sqrt{2}\). Observe that the maximum distance separating \(a\) and \(d\) occurs when \(a = r\) and \(d = q\). Then we have that \(|ad| \leq |r| = \sqrt{2}\) and \(|q| = \sqrt{2}\sin(15^\circ) = 2x\sin(15^\circ)\). (This follows immediately from the Law of Sines applied on the right triangles obtained by bisecting isosceles triangles \(\triangle brc\) and \(\triangle cbq\), respectively.) This simplifies to \(|r| = x\sqrt{2}\), therefore \(|ad| \leq |ab|\). If \(|ad| < |ab|\), then \(T\) is not a minimum spanning tree. We observe that \(|ad| = |ab| = x\sqrt{2}\) is impossible, since \(|ad| = x\sqrt{2}\) only when \(a = r, d = q\), and \(|ab| = x\sqrt{2}\) only when \(a = k \neq r\).

So assume without loss of generality that \(a\) lies above the line \(\ell_{rb}\); the case when \(d\) lies below the line \(\ell_{rb}\) is symmetric. If \(a\) lies above \(\ell_{rb}\), then \(d\) must lie above \(\ell_{rc}\), otherwise Equation (1) would not hold. We discuss two cases, depending on the relative sizes of \(bc\) and \(cd\).

Case \(|cd| \geq x\). By Property (P2), \(|cd| \leq |db|\), therefore \(b\) lies outside \(D(d,|cd|)\).

Then there is a point \(e\) at the intersection between \(\partial D(c,x)\) and \(\partial D(d,|cd|)\), on the same side of \(\mathcal{H}\) as \(b\). If \(d\) lies on \(\mathcal{H}\), then \(e\) and \(b\) coincide. We assume first that \(d\) lies strictly below \(\mathcal{H}\), so \(e \neq b\) and \(|cd| < |db|\). Note that \(a\) must lie outside \(D(d,|cd|)\); otherwise, \(|da| < |dc|\), so replacing \(dc\) by \(da\) would result in a spanning tree of smaller weight, contradicting the fact that \(T\) is a minimum spanning tree.

If \(ba\) does not cross \(D(d,|cd|)\), then property (P3) implies that \(\angle abc + \angle bcd \geq \angle ebc + \angle bcd\). We now show that \(\angle ebc + \angle bcd > 150^\circ\), contradicting assumption (1). Because \(|dc| = |de| \geq |ce| = |eb|\), we have that \(\angle ecd \geq 60^\circ\). Because \(|eb| = |ec|\), \(\angle bce = 180^\circ - 2\angle bce\). Summing up, we get \(\angle ebc + \angle bcd = 180^\circ + \angle ecd - \angle ebc \geq 240^\circ - \angle ebc > 150^\circ\), because \(\angle ebc < 90^\circ\).

We now show by contradiction that \(ba\) does not cross \(D(d,|cd|)\). Assume the contrary, and let \(d'\) be the orthogonal projection of \(d\) on the horizontal line passing through \(b\). (Refer to left of Figure 4.) We first verify that, when \(d\) lies outside \(D(c,x)\), we have that \(\angle bdd' < 45^\circ\). To see this, observe that \(\angle bdd'\) is maximized when \(d\) is at the intersection of \(D(c,x)\) and \(rc\), in which case \(\triangle dcb\) is isosceles with its apex \(75^\circ\) and the other two angles (and in particular \(\angle ceb\)) each \(52.5^\circ\). Thus the maximum value for \(\angle bdd'\) is \(90^\circ - 2 \times 52.5^\circ = 37.5^\circ < 45^\circ\).
With this, we have that $|dd'| < |d'b|$. Let $w$ be the intersection of $ba$ and $dd'$. 
(Note that, if $\angle abc = 90^\circ$, then $w$ and $d'$ coincide.) The triangle inequality 
applied on triangles $\triangle dwa$ and (eventually degenerate) $\triangle d'wb$ tells us that 
$|ad| < |aw| + |wd|$ and $|bd'| \leq |bu| + |wd'|$. Summing up, we get $|ad| + |bd'| < 
|ab| + |dd'|$. This along with $|dd'| < |d'b|$ implies $|ad| < |ab|$, contradicting the 
fact that $T$ is a minimum spanning tree.

If $d$ lies on $H$, then $b$ and $e$ coincide, and $ba$ must cross $D(d, |cd|)$. Then the 
arguments above go through, leading to $|ad| < |ab|$, contradicting the fact that $T$ is a minimum spanning tree.

Case $|cd| < x$. By Property (P3), $d$ also lies outside $D(b, x)$, therefore $b$ lies 
outside $D(d, x)$. Then there is a point $e$ at the intersection between $\partial D(c, x)$ 
and $\partial D(d, x)$, on the same side of $H$ as $b$. (See Figure 5.) Arguments similar to 
the earlier ones show that $a$ must lie outside $D(d, x)$.

If $ab$ does not cross $D(d, x)$, property (P3) implies that $\angle abc + \angle bcd \geq \angle ebc + 
\angle bcd$. We now show that $\angle ebc + \angle bcd > 150^\circ$, contradicting assumption (1).
Because $|ce| = |de| > |cd|$, we have that $\angle ced > 60^\circ$. From here, arguments identical to the ones used in the previous case show that $\angle ebc + \angle bcd > 150^\circ$, a contradiction.

We now show by contradiction that $ab$ may not cross $D(d, x)$. Assume the contrary, and let $bb'$ be horizontal, with $|bb'| = x$. (Refer to Figure 5.) Then $\angle bqb'$ is equilateral, and $\angle qdb'$ is obtuse. It follows that $|db'| < |qb'| = |bb'|$. This along with the triangle inequality applied twice on the quadrilateral $ab'bd$ implies $|ad| < |ab|$, contradicting the fact that $T$ is a minimum spanning tree. This concludes the proof.

Proof of Lemma 2

**Lemma 2.** Let $(a, b, c, d)$ be a path in $\text{MST}_5$ such that $a$ and $d$ lie on or to the same side of the line $bc$. Furthermore, $60^\circ \leq \angle abc \leq 150^\circ$, $60^\circ \leq \angle bcd \leq 150^\circ$, and $\angle abc + \angle bcd \leq 210^\circ$. Then $|ad| \leq \sqrt{3}$.

**Proof.** Let $D(p, r)$ denote the open disk of radius $r$ centered at point $p$, let $\partial D(p, r)$ denote its boundary, and let $D[p, r] = D(p, r) \cup \partial D(p, r)$ denote the closed disk. Rotate $\text{MST}_5$ so that $bc$ is vertical (as shown in Fig 6), and $a$ and $d$ lie to the right of $bc$. As shown previously in Lemma 1, $a$ lies on or above the perpendicular bisector of $bc$ and $d$ lies on or below it. Thus we may, for simplicity, let $|bc| = 1$, since this is the value for which $|ad|$ is maximum. Assume without loss of generality that $\angle bcd \leq \angle abc$. (The case when $\angle bcd \geq \angle abc$ is symmetrical.) Then $2\angle bcd \leq \angle abc + \angle bcd \leq 210^\circ$, so we are in the situation where

$$\angle bcd \leq 105^\circ$$

(2)

We start by identifying regions of diameter $\sqrt{3}$ that may contain $a$ and $d$.

![Figure 6: Lemma 2: defining points $p$, $q$, $s$ and regions $R_a$ and $R_d$; $R_a \cup R_d$ has diameter $\sqrt{3}$.](image)

Fix $0^\circ < \alpha \leq 30^\circ$, and define the points $p = p(\alpha)$, $q = q(\alpha)$ and $s = s(\alpha)$ as follows: $p$ is the point to the right of $bc$, such that $\angle pbc = 120^\circ + \alpha$ and $|bp| = 1$; $q$ is the right intersection point between $\partial D(p, \sqrt{3})$ and $\partial D(b, 1)$; and $s$ is the
right intersection point between \( \partial D(p, \sqrt{3}) \) and \( \partial D(c, 1) \). Refer to Figure 6.

We also define two closed regions \( R_a = R_a(\alpha) \) and \( R_d = R_d(\alpha) \) as follows: \( R_a \)
is the closed region bounded by \( bp, \partial D(b, 1) \) and \( \partial D(c, 1) \), inside \( D[b,1] \) but
outside \( D(c,1) \); and \( R_d \) is the closed region bounded by \( \partial D(p, \sqrt{3}) \), \( \partial D(b,1) \)
and \( \partial D(c,1) \), inside \( D[c,1] \cap D[p,\sqrt{3}] \) but outside \( D(b,1) \). Next we show that
the following three properties hold for any \( 0^\circ < \alpha \leq 30^\circ \):

(P1) \( \angle pbc + \angle bcq = 210^\circ + \frac{\pi}{2} \).

(P2) The closed region \( R_a \cup R_d \) has diameter \( \sqrt{3} \).

(P3) Point \( a \) lies in \( R_a \). Point \( d \) lies either in \( R_d \) or to the right of the ray \( \overrightarrow{cq} \).

To see why (P1) holds, note that \( \angle pbq = 120^\circ \) (because \( |pb| = |bq| = 1 \) and
\( |pq| = \sqrt{3} \)). It follows that \( \angle cbq = \alpha \). This along with the fact that triangle
\( \triangle bcq \) is isosceles (because \( |bc| = |bq| = 1 \)) implies that
\[ \angle bcq = 90^\circ - \frac{\alpha}{2} \tag{3} \]

Summing up this quantity with \( \angle pbc = 120^\circ + \alpha \) yields property (P1). Property
(P2) follows immediately from the fact \( D[p, \sqrt{3}] \) encloses both \( R_a \) and \( R_d \). We
now turn to property (P3). The following three observations show that \( a \) lies in
\( R_a \), thus establishing the first part of property (P3): (i) \( a \) lies to the right of the
polygonal line \( (p, b, c) \), by the lemma statement (ii) \( a \) lies inside \( D[b,1] \), because
\( |ab| \leq 1 \) by the property of MST\(_5\), and (iii) \( a \) lies outside \( D(c,1) \), because
otherwise \( |ca| < 1 = |bc| \), meaning that \( bc \) could be replaced by \( ac \) to yield a
tree lighter than MST\(_5\), a contradiction. Similar observations show that \( d \) must
lie inside \( D[c,1] \) and outside \( D(b,1) \). Thus \( d \) lies either in \( R_d \) or below the
circular path \( (c,q,s) \) formed by the circular arcs \( (c,q) \in \partial D(b,1) \) and \( (q,s) \in
\partial D(p,\sqrt{3}) \). We now show that \( \angle bcq < \angle bcs \), which along with the arguments
above establishes the second part of property (P3). Note that the quadrilateral
\( \triangle bcs \) has edge lengths \( |pb| = |bc| = |cs| = 1 \), \( |ps| = \sqrt{3} \) and angle \( \angle pbc = 120^\circ + \alpha \).

From the isosceles triangle \( \triangle pbc \) we derive \( \angle bcp = 30 - \alpha/2 = \angle bcs - \angle pcs \).
This along with equality (3) implies that \( \angle bcq \leq \angle bcs \) holds if \( \angle pcs \geq 60^\circ \),
so we focus on proving the latter. The Law of Sines applied on triangle \( \triangle pbc \)
yields \( |pc|/\sin(120^\circ + \alpha) = 1/\sin(30^\circ - \alpha/2) \). Simple calculations involving the
substitution \( \sin(120^\circ + \alpha) = \sin(60^\circ - \alpha) = 2\sin(30^\circ - \alpha/2)\cos(30^\circ - \alpha/2) \)
yield \( |pc| = 2\cos(30^\circ - \alpha/2) \). We can now plug this quantity into the equality
\( |ps|^2 = |pc|^2 + |cs|^2 - 2|pc||cs|\cos(\angle pcs) \) (which holds by the Law of Cosine on
triangle \( \triangle pcs \)) to obtain (after a few simplifications)
\[
\cos(\angle pcs) = \frac{\cos(60^\circ - \alpha)}{2\cos(30^\circ - \alpha/2)}
\]
(We used the substitution \( \cos(60^\circ - \alpha) = 2\cos^2(30^\circ - \alpha/2) - 1 \) in deriving the
equality above.) It can be easily verified that the right side of the equality
above does not exceed 1/2 for any \( 0 \leq \alpha \leq 30^\circ \). It follows that \( \angle pcs \geq 60^\circ \),
thus settling property (P3).
Having established that properties (P1), (P2) and (P3) hold for any \( \alpha \) value in the interval \((0^\circ, 30^\circ]\), we use these properties to show that for any \( a \) such that \( \angle abc = 120^\circ + \alpha \in (120^\circ, 150^\circ] \), the lemma is satisfied. Let points \( p, q, s \) be as defined above, and similarly regions \( R_a \) and \( R_d \). (See Figures 7a for \( \alpha = 30^\circ \) and 7b for \( \alpha = 15^\circ \).) By property (P3), \( a \in R_a \). If \( d \in R_d \), then the lemma holds by property (P2). Suppose for contradiction that \( d \notin R_d \).

![Figure 7: Proof of Lemma 2](image)

(a) (b) (c)

(\( \alpha = 30^\circ \)) (Intermediate case: \( \alpha = 0^\circ \)) (Extreme case: \( \alpha = 15^\circ \))

Then by property (P3), \( \angle bcd > \angle beq \). Because \( \angle abc + \angle bcd \leq 210^\circ \), we have \( \angle abc \leq 210^\circ - \angle bcd < 210^\circ - \angle beq \). By property (P1), \( \angle beq = 210^\circ + \alpha/2 - \angle pbc \). Thus \( \angle abc < 210^\circ - (210^\circ + \alpha/2 - \angle abc) \), noting that \( \angle abc \leq \angle pbc \). This simplifies to \( \angle abc < \angle abc - \alpha/2 \), which is a contradiction for \( \alpha \geq 0 \). Therefore \( d \in R_d \).

We now consider the remaining cases when \( \angle abc \in [60, 120] \), or stated in terms of \( \alpha \), \( \angle abc = 120^\circ + \alpha \), for some \( \alpha \in [-60^\circ, 0^\circ] \). Let points \( p, q, s \) be defined in terms of \( \alpha \) as before, and similarly regions \( R_a \) and \( R_d \). For this \( \alpha \) interval, observe that property (P2) holds, as does the following modified version of property (P3):

(P3') Point \( a \) lies in \( R_a \). Point \( d \) lies either in \( R_d \) or to the right of the ray \( \overrightarrow{cs} \).

Consider first \( \angle abc = 120^\circ + \alpha \), for \( \alpha = 0 \). By property (P3'), \( a \in R_a \). If \( d \in R_d \), then the lemma holds by property (P2). So suppose for contradiction that \( d \notin R_d \). Observe that when \( \alpha = 0 \), \( q \) and \( c \) coinide, \( \angle pbc = 120^\circ \), and \( |pc| = \sqrt{3} \). (See Figure 7c.) Because \( \triangle pbc \) is isosceles, \( \angle bcp = 30^\circ \). By definition \( |ps| = \sqrt{3} \) and \( |cs| = 1 \). Applying the Law of Cosines to \( \triangle pcs \) yields \( \cos(\angle pcs) = 1/(2\sqrt{3}) \), so \( \angle pcs > 72^\circ \). Therefore \( \angle bcs > 102^\circ \). By property (P3'), \( d \) lies to the right of \( \overrightarrow{cs} \), and so \( \angle bcd \geq \angle bcs > 102^\circ \). This leads to the contradiction that \( \angle abc + \angle bcd > 222^\circ \). Observing that \( \angle bcs \) increases as \( \alpha \) decreases, the same argument above also proves the lemma for all \( \alpha \in [0^\circ, -12^\circ] \).

Before continuing, we establish some facts for \( \alpha = -12^\circ \). In this case, we have \( \angle abc = \angle pbc = 108^\circ \), and \( \angle bcp = 36^\circ \). Applying the Law of Cosines to \( \triangle pbc \) yields \( |pc| = \sqrt{2 - 2 \cos 108^\circ} \). Applying the Law of Cosines to \( \triangle pcs \) yields...
\[
\cos \angle pcs = 2 \cos 108/(-2|pe|), \text{ and so } \angle pcs > 78^\circ. \text{ Thus } \angle bcs > 114^\circ. \text{ Because } \\
\angle bcs \text{ increases as } \alpha \text{ decreases, } \angle bcs > 114^\circ \text{ for all } \alpha \in [-12^\circ, -60^\circ].
\]

We complete the proof by consider points \( a \) such that \( \angle abc = 120^\circ + \alpha \), for \( \alpha \in [-12^\circ, -60^\circ] \). In these cases, if \( d \in R_d \), then the lemma holds, so assume \( d \notin R_d \). By Property (P3'), \( \angle bcd \geq \angle bcs \), which we established above is greater than 114°. But this contradicts inequality (2).