Rotational Clamshell Casting In Two Dimensions

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Abstract

A popular manufacturing technique is *clamshell casting*, where liquid is poured into a cast and the cast is removed once the liquid has hardened. We consider the case where the object to be manufactured is a simple polygon with n vertices in the plane. The cast consists of exactly two parts and is removed by a rotation around a point in the plane. The following two problems are addressed: (1) Given a cast and a center of rotation r in the plane, we determine in $O(n)$ time whether there exists a partitioning of the cast into exactly two parts, such that one part can be rotated clockwise around r and the other part can be rotated counterclockwise around r without colliding with the interior of the polygon. (2) An algorithm is presented to find all the points in the plane that allow a cast partitioning as described above. For convex polygons, an algorithm with running time $O(n)$ is presented. For simple polygons, the algorithm's running time becomes $O(n^2)$.

1 Introduction

The problem of whether a given object modeled by a polygon can be manufactured using the casting process is a well-known problem in computational geometry. In fact, the problem is discussed in Chapter 4 of the text book by de Berg et al. [7]. The following overview of related problems is not extensive. For a detailed discussion of problems related to manufacturing processes considered in computational geometry, the reader is referred to Bose [3], Bose and Toussaint [6], and the Handbook of Discrete and Computational Geometry by Goodman and O'Rourke [10].

The geometric setting of *clamshell casting* is considered in two dimensions. In the following, we explain what we mean by clamshell casting.

Assume that we wish to manufacture an object modeled by a simple polygon P with n vertices. Let the boundary of P be the cast of P. Two problems are addressed. First, given a center of rotation r in the plane, determine whether there exists a partitioning of the cast into exactly two parts, such that one part can be rotated clockwise around r and the other part can be rotated counterclockwise around r without colliding with the interior of P. We present an algorithm to solve this problem with running time $O(n)$. Second, an algorithm is presented to find all the points in the plane that allow a cast partitioning as described above. The algorithm's running time for convex polygons is $O(n)$. For simple polygons with reflex vertices, the algorithm requires time $O(n^2)$. We provide an $\Omega(n^2)$ lower bound thereby proving the optimality of the algorithm.

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There exists a close relationship between rotational casting and translational casting. Let P denote a polygon in 2-dimensional space and let r denote the center of rotation used to rotationally cast P. Assume that P and r are given in a polar coordinate system with origin r. In polar coordinates, every point is described by an angle ϕ and the distance d from the origin. Transform the coordinate system into a cartesian coordinate system, such that the x-axis describes the angle ϕ and the y-axis describes the distance d. Considering this transformed system between $x = 0$ and $x = 2\pi$ shows the transformed polar coordinate system. Every point of the cast of P moves along a straight line when the cast is removed. This means that rotational casting becomes translational casting. It remains to analyze the shape of the transformed polygon $\tilde{P}$. Without loss of generality (since everything can be rotated), assume that P does not contain vertical edges. Furthermore, assume that r is not contained in the interior of P, since otherwise, P could not be cast using r as center of rotation. The transformation of an edge e of P is in essence a curve describing the distance of points on the edge to the origin. The distance from an edge to a point is non-linear and can not be described using an algebraic curve, but trigonometric functions are necessary [2]. Since r is not contained in the interior of P, the transformed polygon $\tilde{P}$ is topologically equivalent to P and its boundary consists of piecewise non-algebraic curves. Considering rotational casting of a polygon P is equivalent to considering translational casting of the transformed polygon $\tilde{P}$. This relationship between rotational casting and translational casting extends to three dimensions.

To our knowledge, this close relationship between rotational casting and translational casting has not previously been mentioned or used to obtain algorithms to rotationally cast polygons or polyhedra. None of the algorithms surveyed in the following can easily be extended to handle translational castability of 2-dimensional objects bounded by piecewise non-algebraic curves.

Rosenbloom and Rappaport [16] consider an object modeled by a simple polygon with n vertices and they define the cast of this object as the boundary of the polygon. They published algorithms to solve two related problems. The first algorithm determines in time $O(n)$ whether the cast can be partitioned into exactly two pieces, such that both pieces can be removed from the manufactured object by translation without breaking the object or the cast. To solve this problem, an important link between castability and monotonicity of a simple polygon is established. The algorithm that solves this problem uses the algorithm developed by Preparata and Supowit [15] that determines in $O(n)$ time whether a polygon with n vertices is monotone. The second algorithm determines in time $O(n \log n)$ whether the cast can be partitioned into two pieces by cutting the cast along a line, such that both pieces can be removed from the manufactured object by translation without breaking the object or the cast. If this is possible, the object can be manufactured the following way: the cast is cut into two pieces and the two pieces are put on their sides and filled with liquid. Once the liquid has hardened, the cast is removed by translation and the two manufactured pieces are glued together. This way of manufacturing an object is called sand-casting.

Different approaches exist to examine the three-dimensional version of the casting problem, where the object to be manufactured is modeled by a polyhedron of arbitrary genus and the polyhedron's boundary is used as cast. Ahn et al. [1] determine whether the cast can be partitioned into exactly two pieces, such that both pieces can be removed from the manufactured object by translations in opposite directions without breaking the object or the cast. Bose et al. [4] consider an object modeled by a simple polyhedron and use the polyhedron's boundary as cast. They determine whether the object can be manufactured by sand-casting.

This paper is organized as follows. Section 2 introduces the notation and preliminaries used throughout this paper. Section 3 discusses the problem of finding a partitioning of a given cast based on a given point of rotation, and Section 4 discusses the problem of finding all of the points in the plane that allow a valid partitioning of the cast. Finally, Section 5 concludes and gives ideas for future work.

2 Preliminaries

Let P be a simple polygon in the plane with n vertices and let $int(P)$ and ∂P denote the interior and boundary of P, respectively, so that $P = int(P) \cup \partial P$. The boundary is also called the *cast* of P. The edges of P are oriented in counterclockwise order such that $int(P)$ is located to their left. Parallel adjacent edges are not allowed, since this can be easily avoided by merging the two adjacent parallel edges. The aim is to rotationally remove the cast of P in two pieces. We specify below precisely what this means.

Definition 1. Let r and p be points in the plane. Denote the circular arc with center r and angle α starting at p winding in clockwise (cw) or counterclockwise (ccw) direction by $cwarc(r, p, \alpha)$ or $ccwarc(r, p, \alpha)$ respectively. An edge e of P is *removable in cw orientation with respect to* r if

$$\exists\, \alpha > 0 \textit{ such that } \forall\, p \textit{ on } e : cwarc(r, p, \alpha) \cap int(P) = \emptyset$$

and *removable in ccw orientation with respect to* r if

$$\exists\, \alpha > 0 \textit{ such that } \forall\, p \textit{ on } e : ccwarc(r, p, \alpha) \cap int(P) = \emptyset.$$

Then, we call the cw or ccw orientation a *valid orientation for cast removal for* e *with respect to* r respectively, and we call r a valid center of rotation for e. Figure 1 illustrates the definition of castability for edges.

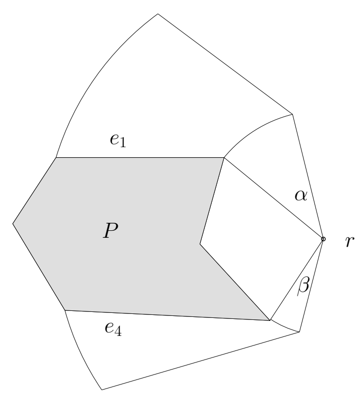

Figure 1: *The edges e_1 and e_4 are removable in cw orientation with angle α and ccw orientation with angle β with respect to r respectively.*

Definition 2. Let r be a point in the plane. A polygon P is *rotationally castable with respect to* r, if ∂P can be partitioned into exactly two connected chains, such that all edges of one chain are removable in cw orientation with respect to r and all edges of the other chain are removable in ccw orientation with respect to r.

Note that the partitioning of the chain is not necessarily at vertices of P. Henceforth, unless stated otherwise, by castable, we will mean rotationally castable. We now outline a key property that characterizes all locations from which an edge is removable.

For an edge $e \in \partial P$ with incident vertices a and b, let $n_e(a)$ denote the line perpendicular to e passing through a. The line $n_e(a)$ divides the plane into two half planes and the open half plane containing b is denoted by $n_e^+(a)$ and the open half plane that does not contain b is denoted by $n_e^-(a)$. The supporting line $l(e)$ of e divides the plane into two half planes. Denote the open half plane located to the left of e when traversing P in ccw orientation by $l^+(e)$ and the open half plane located to the right of e when traversing P by $l^-(e)$, see Figure 2. The closure of an open set S is denoted by $cl(S)$.

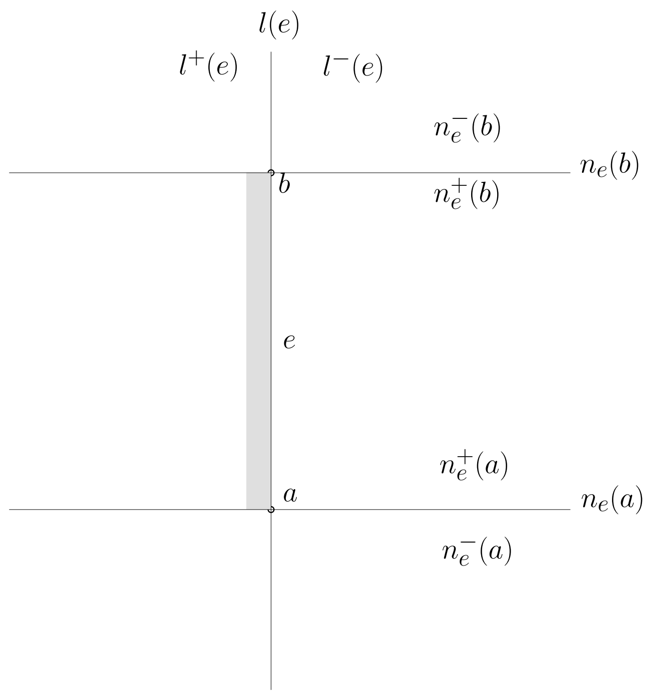

Figure 2: *The half planes associated with an edge e.*

Lemma 1. *Let e be an edge of P and denote the two vertices incident to e in ccw order by a and b. For the orientation for cast removal of e, the following four cases are possible:*

1. *The edge e is removable from the cast using a cw rotation around r, if and only if $r \in cl(n_e^-(a))$.*

2. *The edge e is removable from the cast using a ccw rotation around r, if and only if $r \in cl(n_e^-(b))$.*

3. *The edge e needs to be partitioned into two parts at the orthogonal projection of r on e, if and only if $r \in n_e^+(a) \cap n_e^+(b) \cap cl(l^-(e))$. One part of e is removable using a ccw rotation and the other one using a cw rotation around r. Let r^* be the orthogonal projection of r on e. Denote the edge with incident vertices a and r^* by e_1 and the edge with incident vertices r^* and b by e_2 respectively. The edge e_1 is removable using a ccw rotation around r and e_2 is removable using a cw rotation around r.*

4. *The edge e is not removable from the cast, if and only if $r \in n_e^+(a) \cap n_e^+(b) \cap l^+(e)$.*

Proof. Consider that every point p of e moves on $cwarc(r, p, \alpha)$ or $ccwarc(r, p, \alpha)$ when rotated by an angle α around r. Denote the vector from p to r by $\vec{pr}$ and the vector $\vec{pr}$ rotated in ccw orientation around 90^0 by $\vec{pr}^\perp$. For any p not incident to the perpendicular projection of r on e, infinitesimal movements along $cwarc(r, p, \alpha)$ or $ccwarc(r, p, \alpha)$ correspond to infinitesimal movements along the vector $\vec{pr}^\perp$ or $-\vec{pr}^\perp$ respectively. Hence, only translations need to be considered.

Let p be an arbitrary point in the interior of e. There exists an open disk b with positive radius centered at p with the property that exactly half of b is contained in $int(P)$ and exactly half of b is contained in the exterior of P. Denote the ray starting at p propagating in direction $\vec{pr}^\perp$ by q^+ and denote the ray starting at p propagating in direction $-\vec{pr}^\perp$ by q^-.

Let $r \in cl(n_e^-(a))$ and let p be an arbitrary point in the interior of e. The intersection $b \cap q^+$ is located completely outside of $int(P)$. Hence, p can move infinitesimally along $\vec{pr}^\perp$ without penetrating $int(P)$. Infinitesimal movements along $\vec{pr}^\perp$ correspond to infinitesimal movements along $cwarc(r, p, \alpha)$ and hence, $\exists\ \alpha > 0\ such\ that\ \forall\ p\ on\ e : cwarc(r, p, \alpha) \cap int(P) = \emptyset$. The intersection $b \cap q^-$ is completely contained in $int(P) \cup p$ and hence, p can not move infinitesimally along $-\vec{pr}^\perp$ without penetrating $int(P)$. Since infinitesimal movements along $-\vec{pr}^\perp$ correspond to infinitesimal movements along $ccwarc(r, p, \alpha)$, there is no $\alpha > 0$ such that $\forall\ p\ on\ e : ccwarc(r, p, \alpha) \cap int(P) = \emptyset$. Hence, e is only removable using a cw rotation around r if $r \in cl(n_e^-(a))$.

Let $r \in cl(n_e^-(b))$ and let p be an arbitrary point in the interior of e. The intersection $b \cap q^-$ is located completely outside of $int(P)$. Hence, p can move infinitesimally along $-\vec{pr}^\perp$ without penetrating $int(P)$. Infinitesimal movements along $-\vec{pr}^\perp$ correspond to infinitesimal movements along $ccwarc(r, p, \alpha)$ and hence, $\exists\ \alpha > 0\ such\ that\ \forall\ p\ on\ e : ccwarc(r, p, \alpha) \cap int(P) = \emptyset$. The intersection $b \cap q^+$ is completely contained in $int(P) \cup p$ and hence, p can not move infinitesimally along $\vec{pr}^\perp$ without penetrating $int(P)$. Since infinitesimal movements along $\vec{pr}^\perp$ correspond to infinitesimal movements along $cwarc(r, p, \alpha)$, there is no $\alpha > 0$ such that $\forall\ p\ on\ e : cwarc(r, p, \alpha) \cap int(P) = \emptyset$. Hence, e is only removable using a ccw rotation around r if $r \in cl(n_e^-(b))$.

If $r \in n_e^+(a) \cap n_e^+(b) \cap cl(l^-(e))$, e is divided into two edges at the orthogonal projection r^* of r on e. Denote the edge with incident vertices a and r^* by e_1 and the edge with incident vertices r^* and b by e_2 respectively. As $r \in cl(n_{e_1}^-(r^*))$ and $r \in cl(n_{e_2}^-(r^*))$, e_1 is only removable using a ccw rotation around r and e_2 is only removable using a cw rotation around r.

If $r \in n_e^+(a) \cap n_e^+(b) \cap l^+(e)$, the orthogonal projection r^* of r on e can not be rotationally removed from the cast. This means, there is no $\alpha > 0$ such that $cwarc(r, r^*, \alpha) \cap int(P) = \emptyset$ or $ccwarc(r, r^*, \alpha) \cap int(P) = \emptyset$ respectively. Therefore, e is not removable with respect to r.

This determines the removability of e depending on the location of r in the plane. Hence, the four statements of Lemma 1 follow directly. □

3 Decision Problem

In this section, we address the question of whether a polygon is castable with respect to a given point of rotation and present an algorithm that solves the problem in linear time.

Assume that a polygon P and a center of rotation r are given. The aim is to determine whether P is castable with respect to r. If P is castable with respect to r, the two points on ∂P, where the cast is cut, need to be found.

Definition 3. A *near point c with respect to r* is defined as $c \in \partial P$ with the property that an infinitesimal neighborhood of c on ∂P is completely outside of the open disk centered at r and passing through c. This means there exists a disk b centered at c with a positive radius, such that all points $q \in (\partial P \cap b) \setminus c$ are outside of the closed disk centered at r and passing through c.

Hence, if c is not a vertex, c is the orthogonal projection of r on an edge of P. Therefore, c locally minimizes the distance between the cast and the center of rotation r.

Definition 4. A *far point f with respect to r* is defined as $f \in \partial P$ with the property that an infinitesimal neighborhood of f on ∂P is completely contained in the closed disk centered at r and passing through f. This means there exists a disk b centered at f with a positive radius, such that all points $q \in \partial P \cap b$ are completely contained in the closed disk centered at r and passing through f.

The point f is always a vertex of P that locally maximizes the distance between the cast and the center of rotation r.

Definition 5. Let $p \in \partial P$ be a vertex of P or a point in the interior of an edge of P. If p is located in the interior of an edge, split the edge into two edges at p. The valid orientation for cast removal with respect to r is said to *change* at p if one of the edges adjacent to p is removable in cw orientation and the other edge adjacent to p is removable in ccw orientation with respect to r.

Lemma 2. *The valid orientation for cast removal with respect to r changes at a point $p \in \partial P$ if and only if p is either a near point or a far point with respect to r.*

Proof. The proof consists of two parts. First, we show that the valid orientation for cast removal with respect to r changes at $p \in \partial P$ if p is a near point or a far point with respect to r. At a far point f, an infinitesimally small neighborhood of f is completely contained in the closed disk induced by the circle b centered at r passing through f. Hence, there is a smaller circle concentric to b that passes through two neighboring points of f. As this circle intersects the polygon twice, one intersection point penetrates $int(P)$ when rotated infinitesimally in cw orientation with respect to r and the other intersection point penetrates $int(P)$ when rotated infinitesimally in ccw orientation with respect to r. Hence, it is not possible to remove the cast in the same orientation. Hence, the valid orientation for cast removal changes at f. The proof is similar for near points where b is infinitesimally enlarged. Again, the two intersection points of the enlarged circle with the polygon can only be removed in different orientations with respect to r.

Second, the valid orientation for cast removal with respect to r changes at no other point but a near point or a far point. Assume that the valid orientation for cast removal with respect to r changes at $p \in \partial P$ with p neither a far point nor a near point. Hence, the circle b centered at r passing through p properly intersects P at p, since p neither locally maximizes nor locally minimizes the distance between ∂P and r. If p is not a vertex of P, but situated in the interior of an edge e of ∂P, e is split into two edges at p. Otherwise, p is a vertex of P and there exist exactly two edges adjacent to p. Therefore, every point $p \in \partial P$ has two adjacent edges. As P is a simple polygon, locally it is located completely to the left of the boundary defined by the two edges adjacent to p. Hence, the valid orientation for cast removal with respect to r does not change at p, which contradicts the initial assumption. Therefore, p must be either a near point or a far point for the valid direction of cast removal with respect to r to change. $\square$

Theorem 1. *Given a center of rotation r, a polygon P is castable with respect to r if and only if there exists exactly one near point c with respect to r and exactly one far point f with respect to r on ∂P.*

Proof. The proof consists of two parts. First, we show that P is castable with respect to r if there exists exactly one near point c and exactly one far point f with respect to r. If there exists exactly one near point c and exactly one far point f with respect to r, the point c minimizes the distance between ∂P and r and f maximizes the distance between ∂P and r. Hence, P is completely contained in the closed annulus defined by the two concentric circles centered at r and passing through c and f respectively. The valid orientation for cast removal with respect to r can only change at c and f, and no other point $\in \partial P$ (Lemma 2). Therefore, one part of the polygon can be removed using a cw rotation and the other part can be removed using a ccw rotation if P is cut at c and f.

Second, if P is castable with respect to r then there exists exactly one far point and exactly one near point with respect to r. A castable polygon with respect to r consists of two cast parts, i.e. there are exactly two points $\in \partial P$ where the orientation of cast removal with respect to r changes. By Lemma 2, this implies that there are exactly two near or far points $\in \partial P$. The extreme value theorem implies that there is always at least one local minimum and one local maximum with respect to the distance from r to ∂P (see Chapter 3 in [12]). Therefore, there must exist at least one near point and one far point $\in \partial P$ with respect to r. Hence, there is exactly one near point and one far point with respect to r on a castable polygon with respect to r. $\square$

Theorem 1 allows us to determine whether a polygon is castable given a center point r by testing how many points $p \in \partial P$ are local extrema with respect to the distance between p and r. The polygon is castable iff there is exactly one maximum and one minimum. To do this test in practice, we establish the following observation:

Observation 1. *For a simple polygon P and a point r in the plane, the number of points $c \in \partial P$ that locally minimize the distance between ∂P and r equals the number of points $f \in \partial P$ that locally maximize the distance between ∂P and r.*

Observation 1 holds because P is a simple closed polygon. Hence, the function describing the distance from r to ∂P is continuous and there is always a local minimum between two local maxima and vice versa for continuous functions (see Chapter 3 in [12]).

Hence, it is sufficient to consider local maxima to decide whether a polygon is castable given a center point r. As each far point must be a vertex of P, one can test for multiple local maxima by traversing the polygon's vertices p and computing the distances between p and r. This takes time $O(n)$, if n denotes the number of vertices of P.

Theorem 2. *Given a polygon P with n vertices and a center of rotation r in the plane, we can test in $O(n)$ time whether P is castable with respect to r.*

4 Determining all valid regions for cast removal

In this section, the aim is to find all points r in the plane, such that a given polygon is castable with respect to r.

Definition 6. The union of all points r in the plane with the property that P is castable with respect to r is the *valid region for cast removal of P*. The complement of the valid region is the *invalid region for cast removal of P*.

The aim is to determine all valid regions in the plane for a given polygon P by partitioning the plane into valid and invalid regions for cast removal. Once a query point r is given, it is possible to determine whether r is a valid center of rotation for P by determining whether r is contained in a valid or an invalid region for cast removal. We will see that convex polygons have a valid region that differs significantly from the valid region of non-convex simple polygons.

4.1 Convex polygons

In this section, we consider convex polygons and show that it is possible to find all valid regions for cast removal in linear time. The plane is partitioned into valid and invalid regions for cast removal by constructing the envelope of an arrangement of half lines.

Lemma 1 implies that every edge e with incident vertices a and b given in ccw order on ∂P splits the plane into regions of different valid orientations for cast removal, see Figure 3.

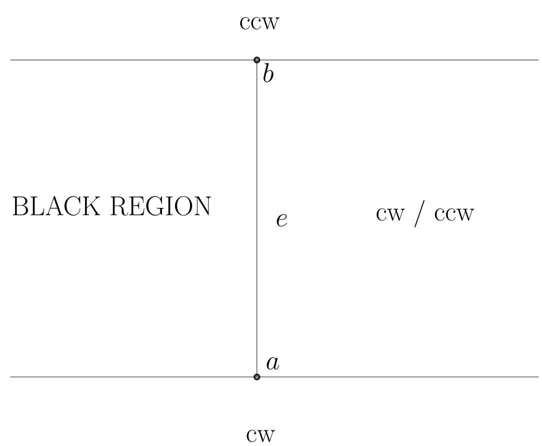

Figure 3: *An edge splits the plane into regions of different valid orientations for cast removal*

Definition 7. Let e be an edge of P and denote the two vertices incident to e in ccw order by a and b. The open strip $n_e^+(a) \cap n_e^+(b) \cap l^+(e)$ is called the *black region of* e.

Note that the black region does not contain any valid centers of rotation r for which e is castable (see Lemma 1, case 4).

Lemma 3. *For a convex polygon P, $int(P)$ is contained in the union of the black regions of the edges of P.*

Proof. Every point $q \in int(P)$ has at least one near point $c \in \partial P$ with respect to q. As P is convex and as $q \in int(P)$, c is the orthogonal projection of q on an edge e and not a vertex of P. Hence, q is contained in the black region of e. □

Lemma 4. *A convex polygon P is castable with respect to a center of rotation r if and only if r is not contained in the union of all black regions of edges of P.*

Proof. This proof consists of two parts. First, a convex polygon is not castable with respect to r if r is contained in the union of all black regions of edges of P. If r is contained in the union of all black regions, it is contained in the black region of at least one edge e. The edge e is therefore not removable with respect to r by Lemma 1.

The second part is that P is castable with respect to r if r is not contained in the union of all black regions of edges of P. Assume, r is outside of the union of black regions, and P is not castable. Theorem 1 and the Extreme Value Theorem [12] imply that there are at least two far points with respect to r. Denote the two far points by f_1 and f_2. Two cases can occur: either $r \in int(P)$ or $r \notin int(P)$. If $r \in int(P)$, Lemma 3 ensures that P is contained in the black region of at least one edge. Hence, $r \notin int(P)$ must hold. Since $r \notin int(P)$, it is possible to compute two tangents from r to ∂P. Denote the two vertices where the tangents touch ∂P by t_1 and t_2, respectively. If a tangent touches ∂P in more than one vertex, choose the vertex closest to r as t_1 or t_2, respectively. The two tangents decompose ∂P into two chains, the lower chain contained in the triangle T with vertices t_1, t_2, and r and the upper chain not contained in T. Since P is convex, no far point of P with respect to r can be on the lower chain. Hence, both f_1 and f_2 are on the upper chain. There are two near points on ∂P with respect to r, one on each chain connecting f_1 and f_2. Since both f_1 and f_2 are on the upper chain, there must be a near point c_1 with respect to r on the upper chain between f_1 and f_2 (see Observation 1). Since P is convex and c_1 is on the upper chain, c_1 can not be a vertex of P. Hence, c_1 is the perpendicular projection of r onto an edge e of P. Since e is on the upper chain and r projects orthogonally onto e, r is located to the left of e. Therefore $r \in n_e^+(a) \cap n_e^+(b) \cap l^+(e)$, where a and b denote the vertices incident to e. This means, r is contained in the black region of e. But this contradicts the initial assumption that r is not contained in the union of all black regions of edges of P. Hence, P is only castable with respect to r if r is outside of the union of black regions of edges of P. □

Lemma 5. *The valid region for cast removal of a convex polygon P consists only of unbounded regions in the plane.*

Proof. Note that Lemma 4 implies that the complement of the union of black regions of edges of a convex polygon P is the valid region for cast removal of P. Assume there exists a point r in a bounded region such that P is castable with respect to r. Then, r is contained in a region bounded by the black regions of at least two edges e_1 and e_2 of P and the convex polygonal chain h connecting e_1 and e_2 that has r to its left, see Figure 4. Let p_1 and p_2 be the vertices $e_1 \cap h$ and $e_2 \cap h$. The vertices p_1 and p_2 minimize the distance from r to e_1 and e_2 respectively. As the function describing the distance from r to ∂P is continuous and as P is simply connected, there exists at least one near point c with respect to r on h. As r is located to the left of h and as h is convex, c is located in the interior of an edge e with incident vertices a and b. Hence, $r \in n_e^+(a) \cap n_e^+(b) \cap l^+(e)$, i.e. r is contained in the black region of e. This contradicts the initial assumption and proves that the valid region of P consists only of unbounded regions in the plane. □

Based on Lemma 4 and Lemma 5, we compute the boundary of the union of all black regions of edges of P. For this, the notion of an envelope of n lines is defined.

Definition 8. A set of n lines in the plane induces a subdivision S of the plane. The *envelope of the n lines* is the polygon formed by the bounded edges of all the unbounded regions of S [11].

Similarly, a convex polygon P and the half lines bounding the black regions of its edges induce a subdivision S of the plane. Parallel half lines with the same orientation intersect at infinity and are therefore considered to be bounded edges. The polygon formed by the bounded edges of all the unbounded regions of S is called *envelope of the arrangement induced by P.*

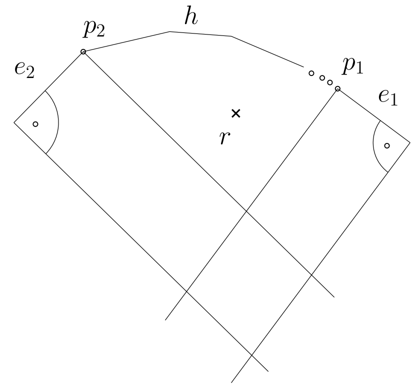

Figure 4: *Location of a point r in a bounded region.*

Lemma 5 implies that all valid regions of P are contained in the complement of the envelope of the arrangement induced by P.

Theorem 3. *Given a convex polygon P with n vertices, a description of the valid regions for cast removal of P has $O(n)$ size and can be computed in $O(n)$ time.*

Proof. Using the algorithm of Keil [11], it is possible to compute the envelope of an arrangement of n lines in $O(n)$ time given that the lines are sorted according to their slope. This algorithm can be modified to find the union of all black regions of edges of P by defining an arrangement consisting of the half lines that bound black regions of edges. In this arrangement, the left and the right envelopes are computed, and their union corresponds to the union of all black regions of P. The modified algorithm first splits the polygon at the two points with minimum and maximum y-coordinate. The right envelope is computed by starting at the lowest point of the polygon and traversing it in clockwise order up to the highest point. For each edge e we traverse, denote the half line in direction of the inner normal of e passing through the first vertex of e encountered during the traversal by l_i and the half line in direction of the inner normal of e passing through the second vertex of e encountered during the traversal by l_i^*, $1 \leq i \leq s, s < n$, see Figure 5. Denote by B_i the convex polygonal chain bounding the region below the half lines l_1 to l_i, $1 \leq i \leq s$, and by A_i the convex polygonal chain bounding the region above the lines l_{i+1}^* to l_s^*, $0 \leq i \leq s-1$. Concatenate A_0, for $1 \leq i \leq s-1$ the boundary of $A_i \cap B_i$, B_s, and in case that A_0 and B_s are disjoint the part of P used to compute the right envelope. For a visualization of the result of this right envelope, refer to Figure 5.

To compute the left envelope, traverse the polygon in ccw starting at the lowest point and ending at the highest point. Define l_i and l_i^*, $1 \leq i \leq s, s < n$ identical to above for every edge of P. Computing A_i and B_i in the same way as before and concatenating A_0, for $1 \leq i \leq s-1$ the boundary of $A_i \cap B_i$, B_s, and in case that A_0 and B_s are disjoint the part of P used to compute the left envelope yields the left envelope. Note that the only difference between this algorithm and Keil's algorithm is the use of two different sets of lines l_i and l_i^* to compute B_i and A_i, respectively. Hence, only minor changes in Keil's algorithm are required to perform these computations. As there are $2n$ half lines already sorted by slope, this algorithm requires $O(n)$ time.

Two planar regions are created, and if we imagine that parallel lines intersect at infinity, the two regions are simply connected planar polygons. The algorithm by Finke and Hinrichs [9], that computes the overlay of simply connected planar subdivisions in time linear in the size of the

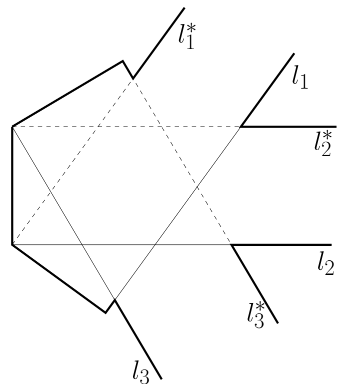

Figure 5: *Result of the right envelope algorithm (shown in fat)*

output, is used to compute the union of those two regions. The algorithm assumes that the two subdivisions are given in quad view data structure and changes that structure in a way that the result represents the overlay of the two regions.

The size of the two envelopes E_1 and E_2 is linear in the number n of vertices of the polygon P, because it can be computed using Keil's algorithm in $O(n)$ time. As both envelopes ordered in clockwise order are given, one can construct a quad view data structure in linear time. The time required for Finke and Hinrichs's algorithm is $O(n + k)$, where n is the combined size of the two polygons to be overlayed and k is the number of intersection points of E_1 and E_2. Lemma 5 guarantees that there are no unbounded valid regions in the overlay of E_1 and E_2. Hence, when an edge of E_1 intersects an edge of E_2, only one of the edges can have further intersection points with E_1 or E_2 respectively. Therefore, the number of intersection points of E_1 and E_2 is $O(n)$ resulting in an $O(n)$ time algorithm. In the resulting subdivision, any region labeled as unbounded is a valid region of P.

The combination of the two algorithms allows to find all valid regions for centers of rotations for cast removal in $O(n)$ time where n is the number of vertices of P. $\square$

Corollary 1. *A convex polygon P with n vertices can be preprocessed in $O(n)$ time, such that for any given point r, we can decide in $O(\log n)$ time if P is castable with respect to r.*

Proof. Theorem 3 allows to find all valid regions for cast removal of P in $O(n)$ time. Hence, in $O(n)$ time, the plane is preprocessed, such that every face of the planar subdivision induced by black regions of P is labeled as valid or invalid region.

For any query point r, after $O(n)$ preprocessing time, it is possible to determine the face of the arrangement containing r in time $O(\log n)$ [13]. Once the face is known, we can determine in constant time whether that face is contained in the union of black regions of P, i.e. whether r is a valid center of rotation. $\square$

4.2 Simple polygons

In this section, we consider simple (not necessarily convex) polygons with n vertices and show that it is possible to find all valid regions for cast removal in $O(n^2)$ time. If the aim is to report all valid regions, this time bound is worst case optimal.

Let r be a point in the plane. If the valid orientation for cast removal of a simple polygon P changes with respect to r at a reflex vertex $v \in \partial P$, v penetrates $int(P)$ when rotated infinitesimally around r with arbitrary orientation. This yields the following observation:

Observation 2. *A simple polygon P cannot be divided at one of its reflex vertices v unless the center of rotation r is v. Hence, v cannot be a far point with respect to r and v can only be a near point with respect to r if $r = v$.*

Definition 9. Let v be a vertex of P and denote the two edges adjacent to v by e_1 and e_2. The *near cone of* v is defined as $cl(n^-_{e_1}(v) \cap n^-_{e_2}(v))$ and denoted by $NC(v)$.

The near cone of v is the set of all points $X \in \mathbb{R}^2$ with the property that v is a near point with respect to X, see Figure 6.

Definition 10. Let v be a vertex of P and denote the two edges adjacent to v by e_1 and e_2. The *far cone of* v is defined as $n^+_{e_1}(v) \cap n^+_{e_2}(v)$ and denoted by $FC(v)$.

The far cone of v is the set of all points $X \in \mathbb{R}^2$ with the property that v is a far point with respect to X, see Figure 6.

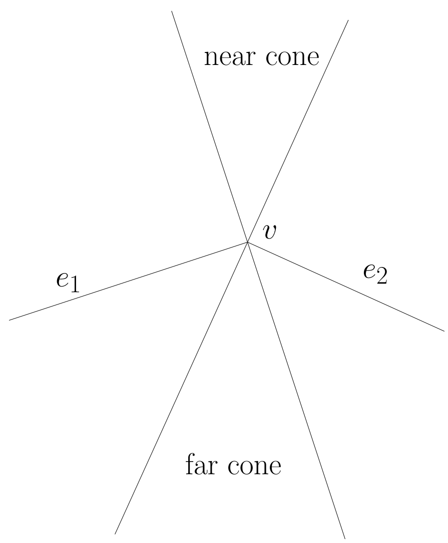

Figure 6: *The near cone and the far cone of v.*

Definition 11. The *black region of a reflex vertex* v is $(NC(v) \cup FC(v)) \setminus v$.

Note that Observation 2 ensures that the black region of v does not contain any valid centers of rotation r that allow to remove v from the cast.

Lemma 6. *For a simple polygon P, $int(P)$ is contained in the union of the black regions of the edges and the reflex vertices of P.*

Proof. Every point $p \in int(P)$ has at least one near point $c \in \partial P$ with respect to p. If c is the orthogonal projection of p on the interior of an edge e, p is contained in the black region of e. Otherwise, c is a reflex vertex and p is contained in the black region of c. $\square$

Lemma 7. *A simple polygon P is castable with respect to a center of rotation r if and only if r is not contained in the union of all black regions of edges and reflex vertices of P.*

Proof. This proof consists of two parts. First, a simple polygon is not castable with respect to r if r is contained in the union of all black regions of edges and reflex vertices of P. If r is contained in the union of all black regions, it is either contained in the black region of at least one edge e or in the black region of at least one reflex vertex v. Hence, either e or v can not be removed from the cast.

Second, a simple polygon is always castable if r is not contained in the union of black regions of its edges and reflex vertices. Assume that P is not castable with respect to r and that r is not contained in the union of black regions of edges and reflex vertices of P. Hence, there are at least two far points f_1 and f_2 on ∂P with respect to r, see Theorem 1 and the Extreme Value Theorem [12]. Note that neither f_1 nor f_2 can be a reflex vertex as r is not contained in the black region of any reflex vertex. Two situations are possible: either $r \in int(P)$ or $r \notin int(P)$. Lemma 6 ensures that $r \notin int(P)$ as any point $q \in int(P)$ is contained in the union of black regions of edges and reflex vertices of P. Denote the far point with smallest distance to r by f_1. If this far point is not unique, choose an arbitrary far point with smallest distance to r. Denote the circle centered at r passing through f_1 by c. In a local neighborhood of f_1, ∂P is contained in the interior of c. However, since f_2 is a far point on ∂P with respect to r with greater or equal distance from r than f_1, ∂P intersects c in at least one point not equal to f_1. Find the first point q_1 of ∂P that intersects c when starting at f_1 and walking along ∂P in ccw orientation. The polygonal chain starting at f_1 and ending at q_1 splits c into two regions. If r is contained in the region of c located to the left of the polygonal chain starting at f_1 and ending at q_1, denote the polygonal chain by *upper chain*. Otherwise, find the first point q_2 of ∂P that intersects c when starting at f_1 and walking along ∂P in cw orientation. By the Jordan Curve Theorem [14], the polygonal chain starting at q_2 and ending at f_1 must be completely contained in the region of c located to the left of the polygonal chain starting at f_1 and ending at q_1. Furthermore, $int(P)$ is contained in the region bounded by the polygonal chain starting at q_2 and ending at f_1 and by the polygonal chain starting at f_1 and ending at q_1. Hence, r is contained in the region of c located to the left of the polygonal chain starting at q_2 and ending at f_1. Denote the polygonal chain starting at q_2 and ending at f_1 by *upper chain*. The points f_1, q_1, and q_2 are points that maximize the distance from the two polygonal chains considered above to r. Hence, by Observation 1 there exists a near point c_1 on the upper chain. Since r is located in the region of c located to the left of the upper chain, c_1 can not be a convex vertex. Hence, c_1 is either located on an edge e of P or c_1 is a reflex vertex of P. If c_1 is located on an edge e, c_1 is the perpendicular projection of r on e and therefore, r is contained in the black region of e. Otherwise, c_1 is a reflex vertex that is a near point and therefore, r is contained in the black region of c_1. Hence, r is either contained in the black region of the reflex vertex c_1 or in the black region of the edge e. But this contradicts the initial assumption that r is not contained in the union of all black regions of edges and reflex vertices of P. Hence, P is only castable with respect to r if r is outside of the union of black regions of edges and reflex vertices of P. □

Theorem 4. *Given a simple polygon P with n vertices, a description of the valid regions for cast removal of P has $O(n^2)$ size and can be computed in $O(n^2)$ time.*

Proof. We preprocess the plane by constructing the full arrangement A of the (full) lines bounding the black regions of edges and reflex vertices. A doubly-connected edge list of the arrangement of n lines has complexity $O(n^2)$ and can be constructed in $O(n^2)$ time, see [8], Chapter 8. Once A

is constructed, each face needs to be labeled as valid or invalid region for cast removal. For this purpose, a boolean value is associated with every edge e and reflex vertex v of P that indicates whether the current location is contained in the black region of e or v respectively. We start at an arbitrary face f of A and test for each edge and reflex vertex of P whether it causes f to be invalid. After testing, we set the boolean value of each edge and reflex vertex appropriately and compute the number b of edges and reflex vertices that cause f to be invalid. Clearly, f is valid if and only if $b = 0$. This computation takes $O(n)$ time as every edge and reflex vertex of P needs to be considered. Next, A is traversed in depth-first order on the graph induced by the vertices and the edges of A. Each time, an edge e_A of A is crossed, we update both the boolean value of the edge or reflex vertex of P that induces e_A and the counter b. This way, every face of A is labeled in constant time a piece. The edge e_A and its incident vertices are valid regions for cast removal if and only if one or more of e_A's adjacent faces is a valid region for cast removal. Hence, A can be labeled in $O(n^2)$ time. □

Corollary 2. *A simple polygon P with n vertices can be preprocessed in $O(n^2)$ time, such that for any given point r, we can decide in $O(\log n)$ time if P is castable with respect to r.*

Proof. Theorem 4 allows to find all valid regions for cast removal of P in $O(n^2)$ time. Hence, the plane is preprocessed, such that every face of the planar subdivision induced by black regions of P is labeled as valid or invalid region in time $O(n^2)$.

For any query point r, after $O(n^2)$ preprocessing time, it is possible to determine the face of the arrangement containing r in time $O(\log n)$ [13]. Once the face is known, the label of the face can be retrieved in constant time. Hence, determining whether r is a valid center of rotation for P takes $O(\log n)$ time. □

We now examine the complexity of the valid regions for centers of rotation. In the best case, i.e. in the case of a convex polygon, the number of valid regions for cast removal is $O(n)$. The number of valid regions can not be $\omega(n^2)$ as the complexity of an arrangement induced by $O(n)$ lines is $O(n^2)$. There exists a class of simple polygons where the number of valid regions is $\Omega(n^2)$. This implies that the $O(n^2)$ time bound is worst case optimal if the aim is to report all valid regions of cast removal for a simple polygon. We now outline the construction of the lower bound.

Consider a simple polygon P consisting of $n = 3s - 1$ vertices located on two different polygonal chains. Let s vertices of P be evenly distributed on the upper half of the unit circle. The coordinates of those vertices are

$$(\cos((i-1)\phi_1), \sin((i-1)\phi_1)), i = 1, \ldots, s,$$

where $\phi_1 = \frac{\pi}{s-1}$. Hence, the vertices form a convex polygonal chain c_1. All valid regions induced by c_1 are cones with apex a on the unit circle and opening angle $\frac{\phi_1}{2}$, see Figure 7.

The second polygonal chain c_2 consists of $2s - 1$ vertices. Let s vertices of c_2 be evenly distributed on the arc of the circle with center $\left(-\frac{1}{2}, 0\right)$ and radius 1 starting at $\frac{3\pi}{2}$ and ending at $\frac{25\pi}{16}$. The coordinates of those vertices are

$$\left(-\frac{1}{2} + \cos\left(\frac{3\pi}{2} + (i-1)\phi_2\right), \sin\left(\frac{3\pi}{2} + (i-1)\phi_2\right)\right), i = 1, \ldots, s,$$

where $\phi_2 = \frac{\pi}{16(s-1)}$. Denote the vertices by $v_1, \ldots, v_s$ and note that v_i is not located in the interior of the unit disk for $i = 1, \ldots, s$. Define the vertices v_0, v_{s+1} as

$$\left(-\frac{1}{2} + \cos\left(\frac{3\pi}{2} - \phi_2\right), \sin\left(\frac{3\pi}{2} - \phi_2\right)\right), \left(-\frac{1}{2} + \cos\left(\frac{3\pi}{2} + s\phi_2\right), \sin\left(\frac{3\pi}{2} + s\phi_2\right)\right),$$

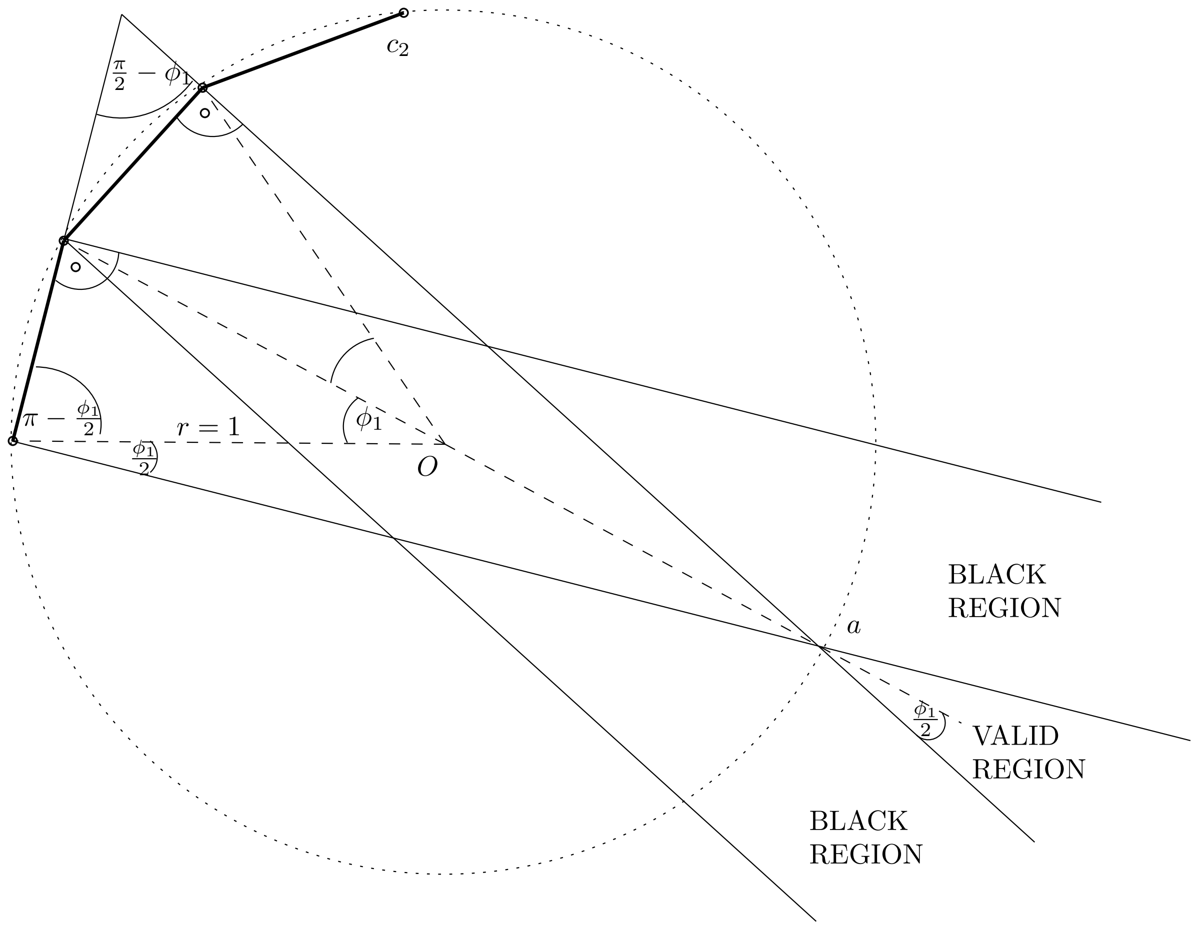

Figure 7: *Approximation of half circle*

respectively. Let $s-1$ vertices of c_2 be defined as the intersections of the line passing through v_{i-1} and v_i with the line passing through v_{i+1} and v_{i+2}, where $i = 1, \ldots, s-1$. These vertices are located on a circle. The polygonal chain c_2 consists of $s-2$ reflex, $s-1$ convex, and 2 boundary vertices. Note that c_2 consists of sides of isosceles triangles, i.e. all the edges have the same length, see Figure 8. Valid regions bounded by part of c_2 and two parallel half lines occur.

The two polygonal chains c_1 and c_2 can now be connected by two edges. This does not introduce further reflex vertices to P, but only two black regions of the new edges. Those black regions have no influence on further considerations. Each of the black regions induced by reflex vertices on c_2 induces a bounded valid region when intersecting the valid region induced by vertices located on the arc of c_1 starting at $\frac{7\pi}{16}$ and ending at $\frac{\pi}{2}$. Hence, there are at least $(s-2)\lfloor \frac{s}{8} \rfloor$ bounded valid regions. As $n = 3s-1$, there are $\frac{n-5}{3}\lfloor \frac{n+1}{24} \rfloor = \Omega(n^2)$ bounded valid regions. Hence, the number of valid regions for cast removal of a simple polygon is $\Omega(n^2)$. An example with $s = 10$ is shown in Figure 9.

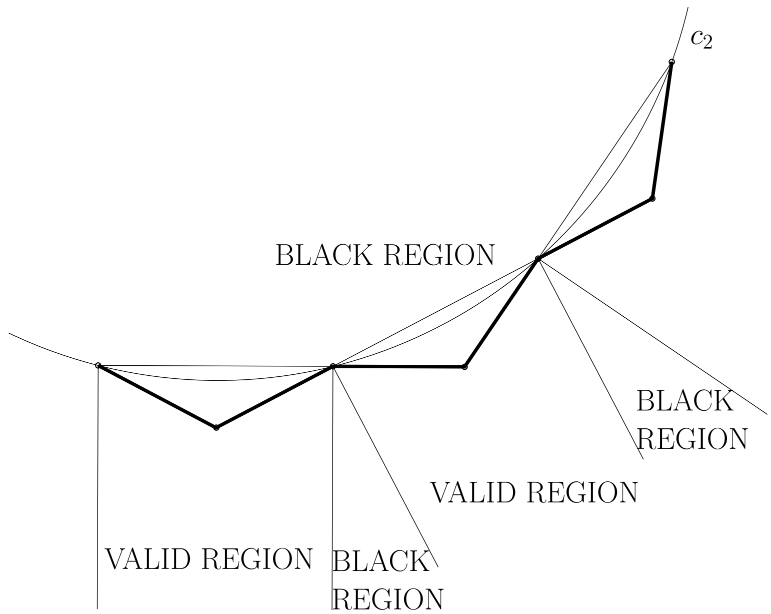

Figure 8: *Polygonal chain*

5 Conclusion and Future Work

We have studied the problem of clamshell casting in two dimensions. An algorithm was developed to solve the problem of determining whether a polygon with n vertices is castable with respect to a given point in the plane with running time $O(n)$. Furthermore, two algorithms were developed to report all the valid centers of rotation for a given polygon in the plane. The running times of the algorithms are $O(n)$ for convex polygons and $O(n^2)$ for simple polygons in general and shown to be worst-case optimal.

The following interesting related problems require further research.

- The extension of the algorithm to three dimensions has recently been published by Bose et al. [5].
- The definition of clamshell casting only tests whether the cast of an object with piecewise linear boundary can be opened by an infinitesimally small angle without breaking the object or the cast. To physically manufacture the object, it is required that the cast can be opened by a sufficiently large angle to remove the object from the cast without breaking the object or the cast. This problem is difficult, since the object can be removed from the cast by an arbitrary sequence of transformations.
- The boundary of the object is defined to be the cast. In case of rotations around infinitesimally small angles, this model is sufficient. However, when considering larger angles of rotations, the thickness of the cast has an influence on the maximum angle of rotation that does not break the object or the cast. Hence, the cast needs to be assigned a thickness.

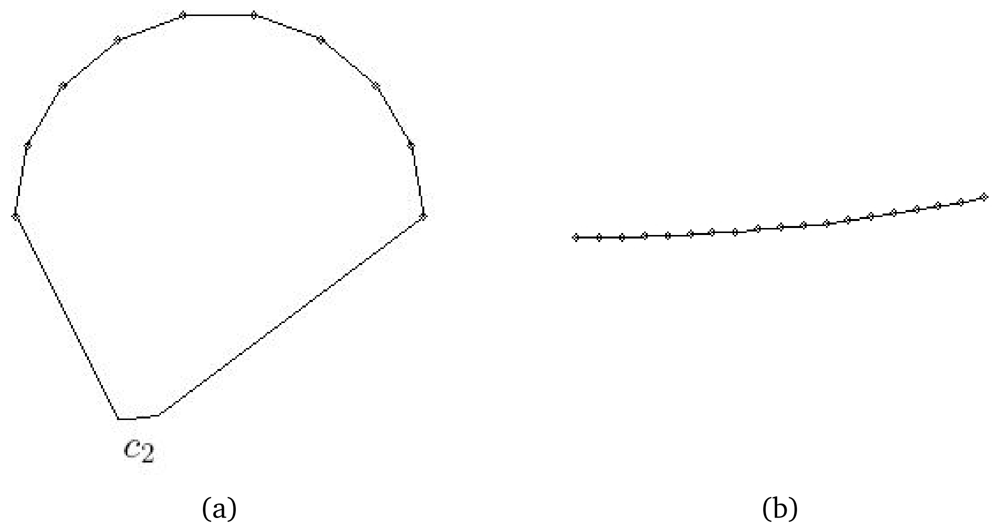

Figure 9: *Example with $s = 10$. (a) shows the polygon, (b) shows an enlargement of the polygonal chain c_2.*

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