Université Grenoble Alpes

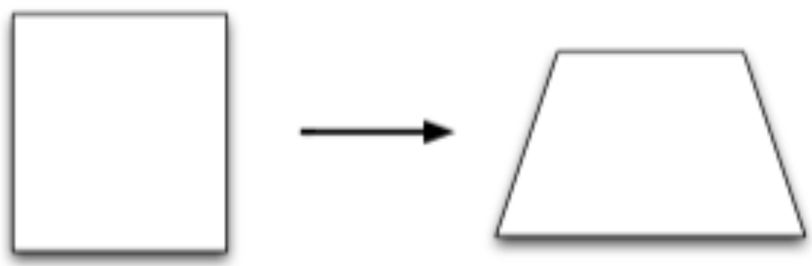
# 3D Vision – Geometry1

Edmond Boyer

# Contents

1. Projective geometry (ref. document available online)
2. Camera model or single view geometry

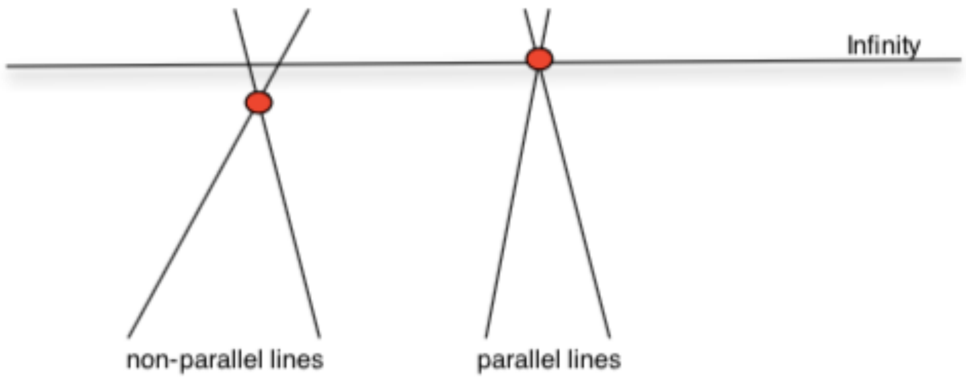
# Projective Geometry

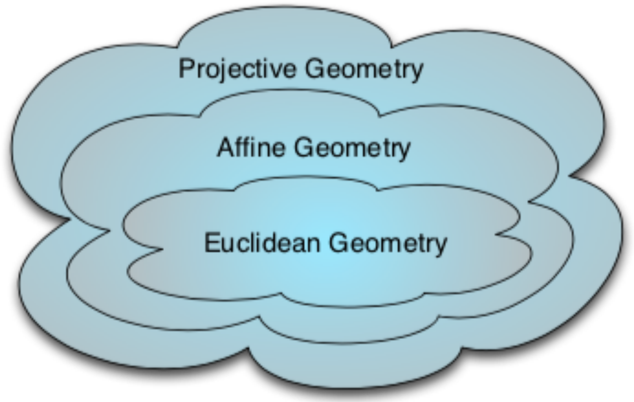


Perspective deformation can be modelled with 2D projective transformation

Projective geometry provides a mathematical formalism to describe the geometry of cameras and the associated transformations -> help designing computational approaches in visual computing.

# Projective Geometry




Projective geometry generalizes definitions and properties, e.g. two lines always interesect, and encompasses affine and Euclidean geometries as subgroups of transformations.

**Key Aspect** -> Infinity is modeled in the projective geometry.

Université Grenoble Alpes

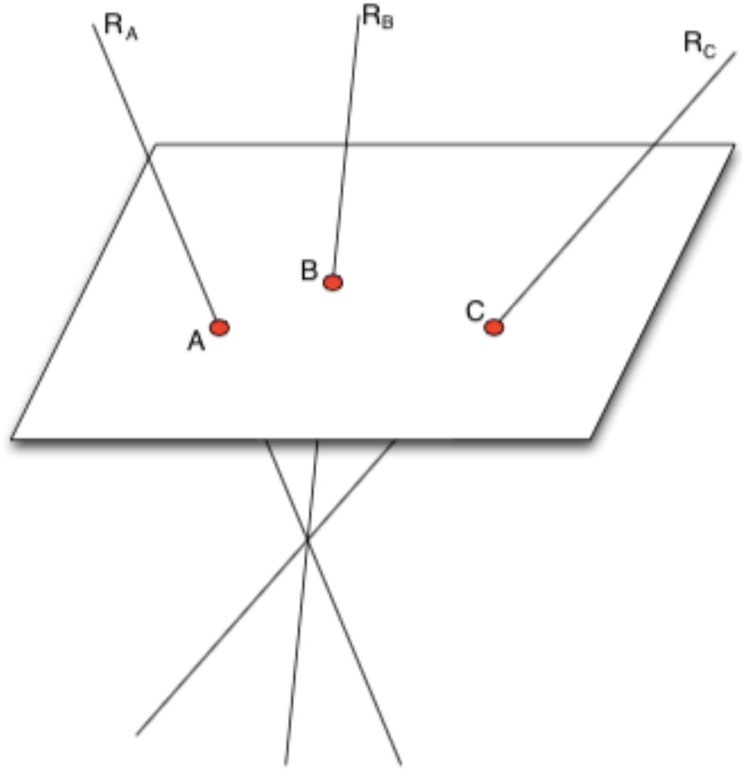
# Projective Geometry - Definitions

**Projective Space:** A point of a real projective space $\mathcal{P}^n$ is represented by a vector of real coordinates $X = [x_0, ..., x_n]^t$ , at least one of which is non-zero. The $\{x_i\}$s are called the projective or homogeneous coordinates and two vectors $X$ and $Y$ represent the same point when there exists a scalar $k \in \mathbb{R}^*$ such that:

$$x_i = k y_i \quad \forall i,$$

which we will denote by:

$$X \sim Y.$$




The projective space $\mathcal{P}^2$ associated to $\mathbb{R}^3$

# Projective Geometry - Definitions

**A projective basis** is a set of $(n+2)$ points of $\mathcal{P}^n$, no $(n+1)$ of which are linearly dependent. For example:

$$\underset{A_0}{\begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ \\ 0 \end{bmatrix}} \quad \underset{A_1}{\begin{bmatrix} 0 \\ 1 \\ . \\ . \\ . \\ \\ 0 \end{bmatrix}} \quad \underset{\dots}{\dots} \quad \underset{A_n}{\begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ \\ 1 \end{bmatrix}} \quad \underset{A^*}{\begin{bmatrix} 1 \\ 1 \\ . \\ . \\ . \\ \\ 1 \end{bmatrix}}$$

is the canonical basis where the $\{A_i\}$s are called the basis points and $A^*$ the unit point.

# Projective Geometry - Definitions

**Projective Transformations:** A matrix $M$ of dimensions $(n+1) \times (n+1)$ such that $det(M) \neq 0$, or equivalently non-singular, defines a linear transformation from $\mathcal{P}^n$ to itself that is called a homography, a collineation or a projective transformation.

*Projective transformations are the most general transformations that preserve incidence relationships, i.e. collinearity and concurrence.*

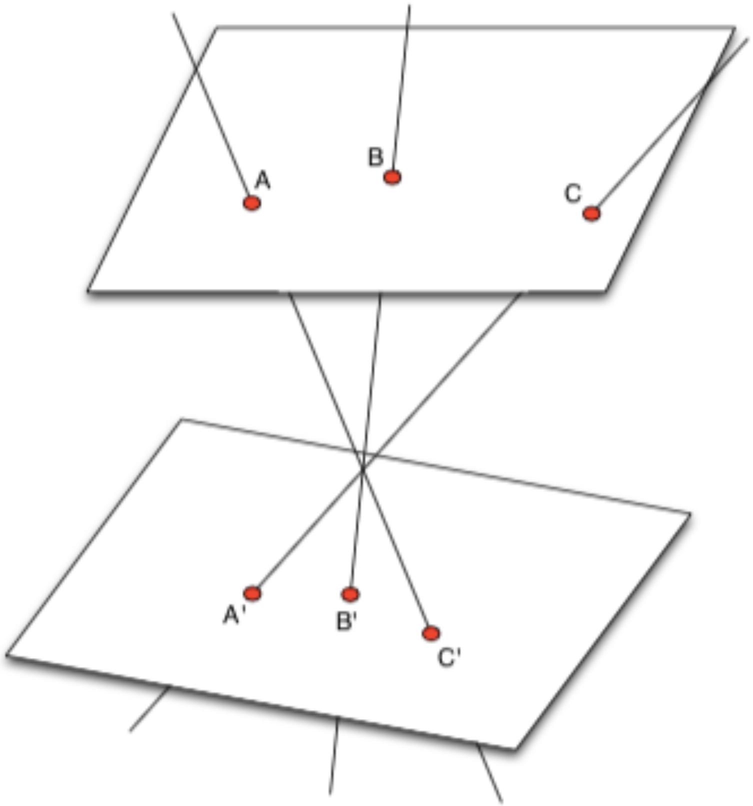
# Projective Geometry - Properties

**Change of basis:** Let $\{X_0, ..., X_{n+1}\}$ and $\{Y_0, ..., Y_{n+1}\}$ be 2 basis of $\mathcal{P}^n$, then there exists a non-singular matrix $M$ of dimension $(n+1) \times (n+1)$ such that:

$$M \cdot X_i \sim Y_i \;\; \forall i,$$

where $M$ is determined up to a scale factor.

*Changes of basis are projective transformations.*



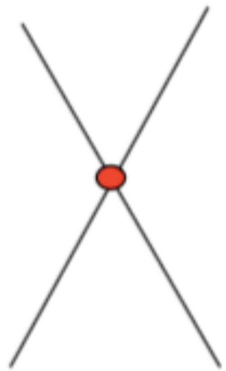

Change of basis in $\mathcal{P}^2$ or projective transformation between $A, B, C$ and $A', B', C'$.

Université Grenoble Alpes

# Projective Geometry - Properties

**Hyperplanes:** Consider $m$ points of $\mathcal{P}^n$ that are linearly independent with $m < n$. The set of points in $\mathcal{P}^n$ that are linearly dependent on these $m$ points form a projective space of dimension $m - 1$. When this dimension is equal to 1, 2 and $n - 1$, this space is called line, plane and hyperplane respectively.

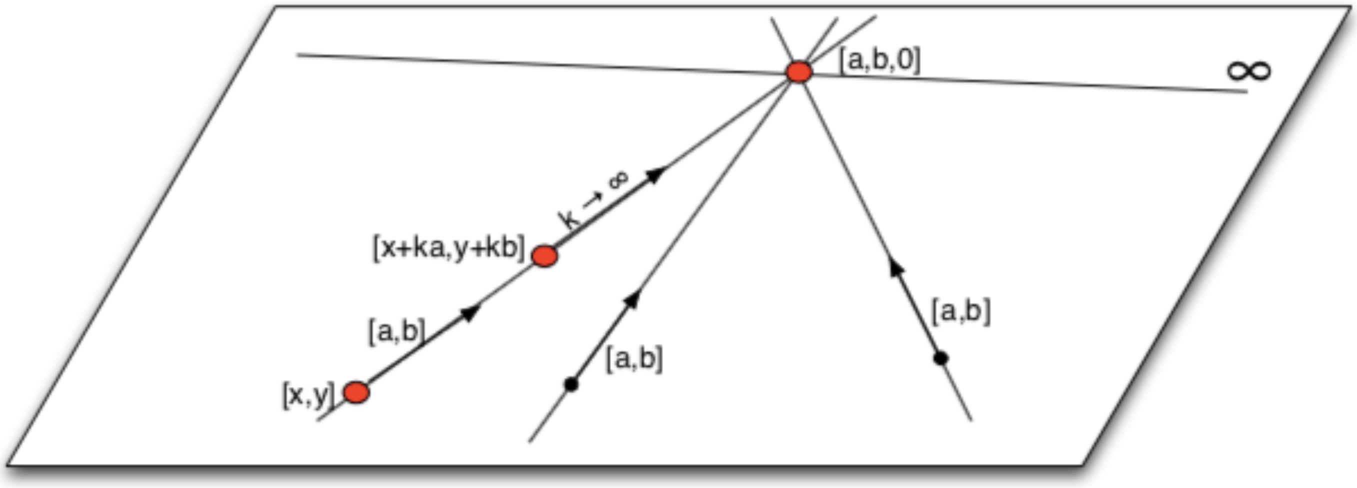
**Duality:** The set of hyperplanes of $\mathcal{P}^n$ is a projective space of dimension $n$. Any definition, property or theorem that applies to the points of a projective space is also valid for its hyperplanes.



2 lines define a point

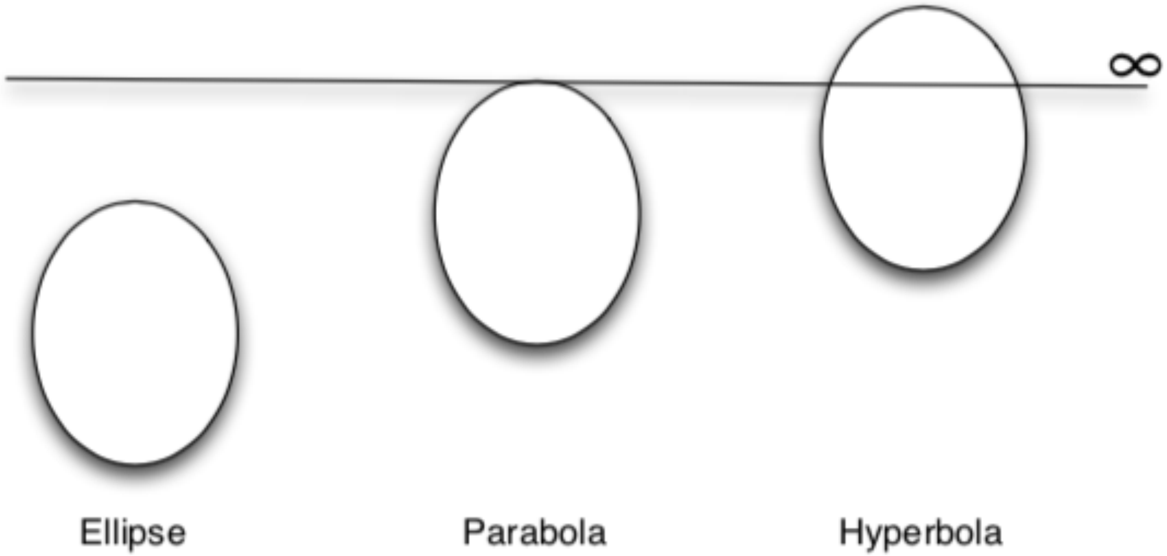
2 points define a line

# Projective Geometry - Properties




**Hyperplane at infinity:** In $\mathcal{P}^2$ any line $L$ is the hyperplane at infinity for the affine space $\mathcal{A}^2 = \mathcal{P}^2 \setminus L$. In this affine space $\mathcal{A}^2$, all lines that share the same direction are concurrent on the line at infinity.

# Projective Geometry - Properties




**Conics:** The affine classification of conics with respect to their incidences with the line at infinity. In the projective plane conics are all projective transformations of the circle

# Projective Geometry – Plane $\mathcal{P}^2$

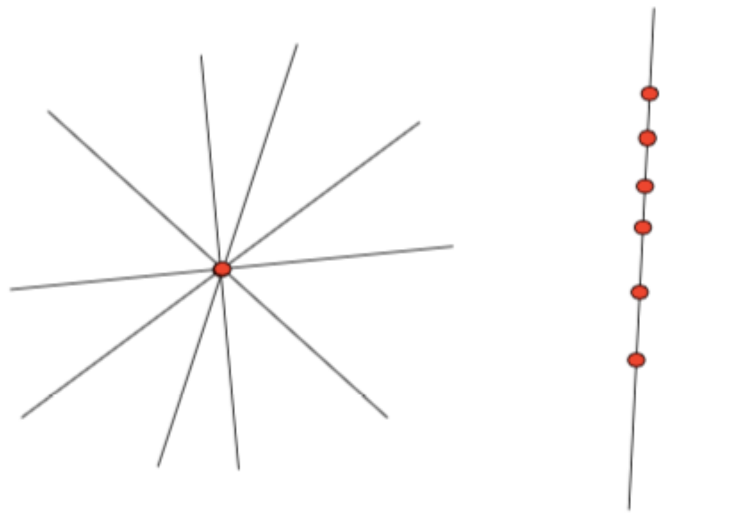
**Points and lines:** points $A = [x_A, y_A, w_A]^t, B = [x_B, y_B, w_B]^t$ and $C = [x_C, y_C, w_C]^t$ are collinear if:

$$\begin{vmatrix} x_A & x_B & x_C \\ y_A & y_B & y_C \\ w_A & w_B & w_C \end{vmatrix} = 0.$$

$$\Leftrightarrow l_{AB} \cdot x_C + m_{AB} \cdot y_C + n_{AB} \cdot w_C = [l_{AB}, m_{AB}, n_{AB}] \cdot \begin{bmatrix} x_C \\ y_C \\ w_C \end{bmatrix}$$

$$\Leftrightarrow L_{AB}^t \cdot C = 0$$

$L_{AB}$ is the line going through $A$ and $B$ and any point $C$ that belongs to $L_{AB}$ statisfy $L_{AB}^t \cdot C = 0$



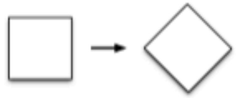
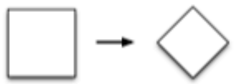
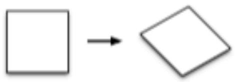
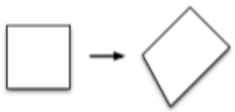
**Duality** $L^t \cdot C$ can describe all lines $L$ going through $C$ or all points $C$ along line $L$.

# Projective Geometry – Plane $\mathcal{P}^2$

## Exercises

1. What is the point $X$ intersection of the lines $L_1$ and $L_2$ ?
2. What is the line $L$ going through the points $X_1$ and $X_2$ ?
3. Consider two lines $L_1 = [l_1, m_1, n_1]^t$ and $L_2 = [l_1, m_1, n_2]^t$, what is the point intersection of these two lines ? what does it represent ?

# Projective Geometry – Plane $\mathcal{P}^2$

| Transf. group | Dof | Matrix | Deformation | Invariants |
|---|---|---|---|---|
| Euclidean | 3 | $\begin{bmatrix} \cos\theta & -\sin\theta & T_0 \\ \sin\theta & \cos\theta & T_1 \\ 0 & 0 & 1 \end{bmatrix}$ |  | length, area |
| Isometry | 4 | $\begin{bmatrix} \epsilon\cos\theta & -\sin\theta & T_0 \\ \epsilon\sin\theta & \cos\theta & T_1 \\ 0 & 0 & 1 \end{bmatrix}$ |  | length ratio, angle, absolut |
| Affine | 6 | $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix}$ |  | parallelism, area ratio, length ratio on a line, linear vector combinations |
| Projective | 8 | $\begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix}$ |  | incidence, collinearity, concurrence, cross-ratio |

Transformation hierarchy

# Projective Geometry – Plane $\mathcal{P}^2$

## Exercises

1. How can we determine a homography $H$ given 4 point correspondences ?
2. Show that if $H$ is a homography that transforms points then the associated transformation for lines is: $H^{-t}$.
3. Show that affine transformations preserve parallelism but not projective transformations.
4. Show that collinearity and concurrence are preserved by projective transformations.

Université Grenoble Alpes

# 3D Geometry

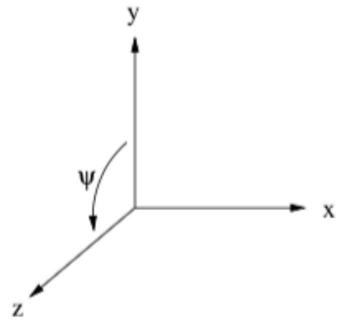
## 3D elementary transformations

Translations: $$T = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

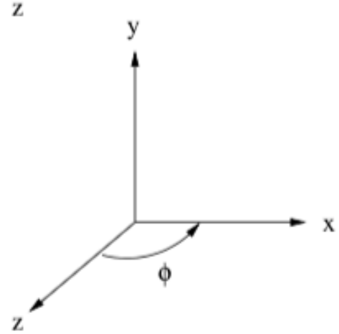
Rotations: $$R = R_z \cdot R_y \cdot R_x,$$

- $\det R = 1,$
- $R^{-1} = R^t.$

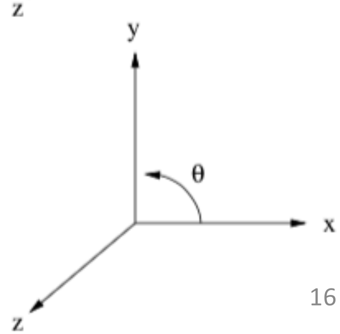
$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi & 0 \\ 0 & \sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$




$$R_y = \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



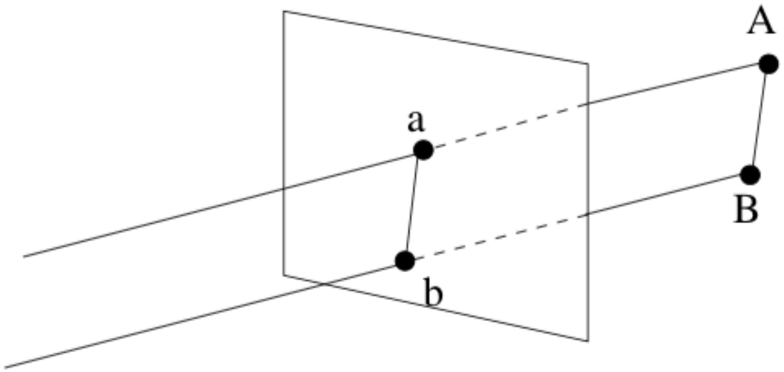

$$R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



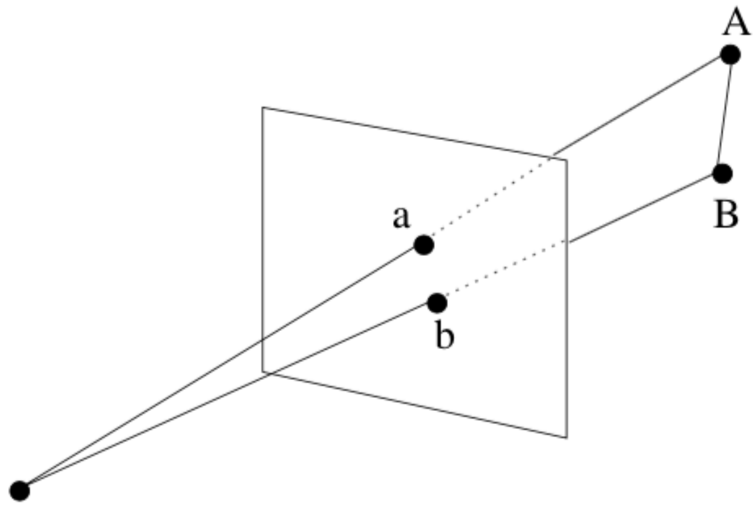
UGA Université Grenoble Alpes

# 3D Geometry

## Projections



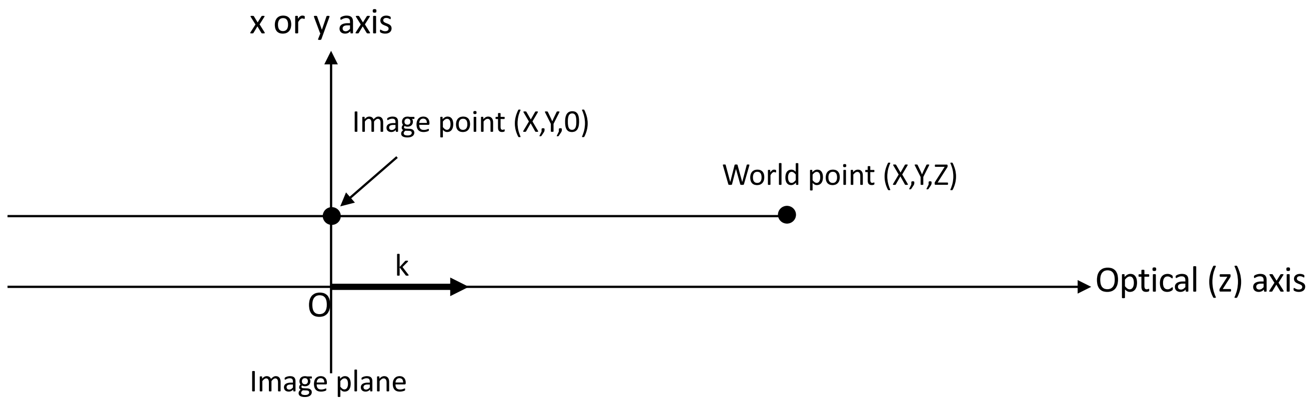

Parallel projections




Perspective projections

# 3D Geometry

## Parallel projections: Orthographic projections



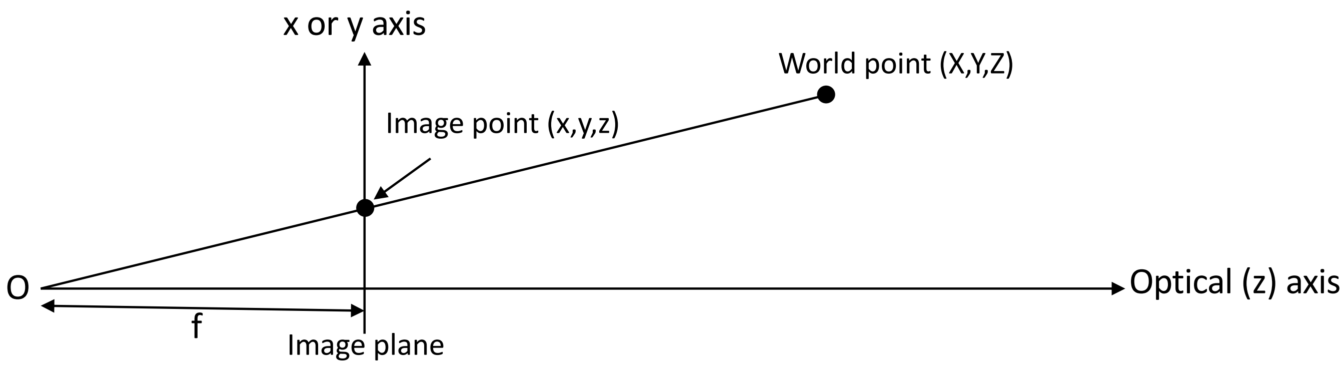

O is the origin, the projection is perpendicular to the image plane

$$\begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Orthographic projection matrix

# 3D Geometry

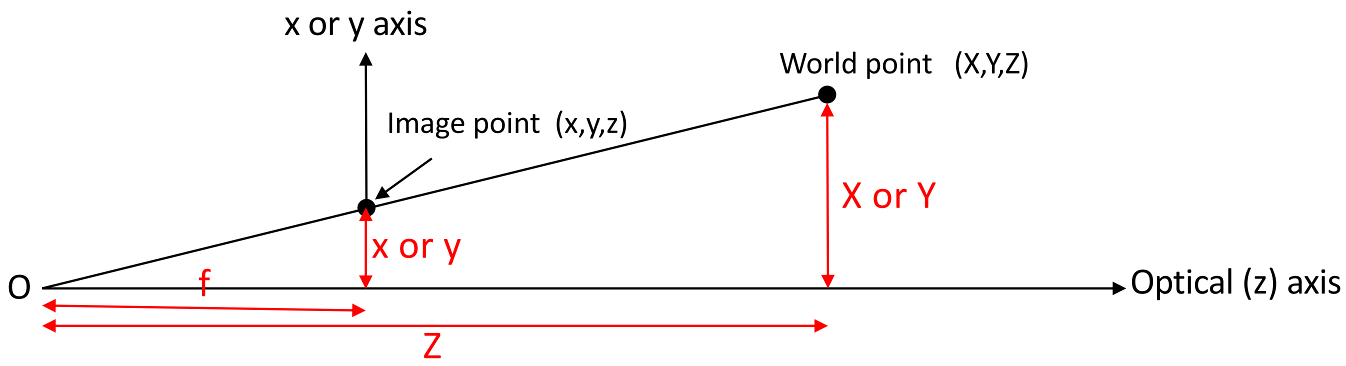
## Perspective projections




O is the projection centre, the optical axis the ray going through O and perpendicular to the image plane, f the focal length.

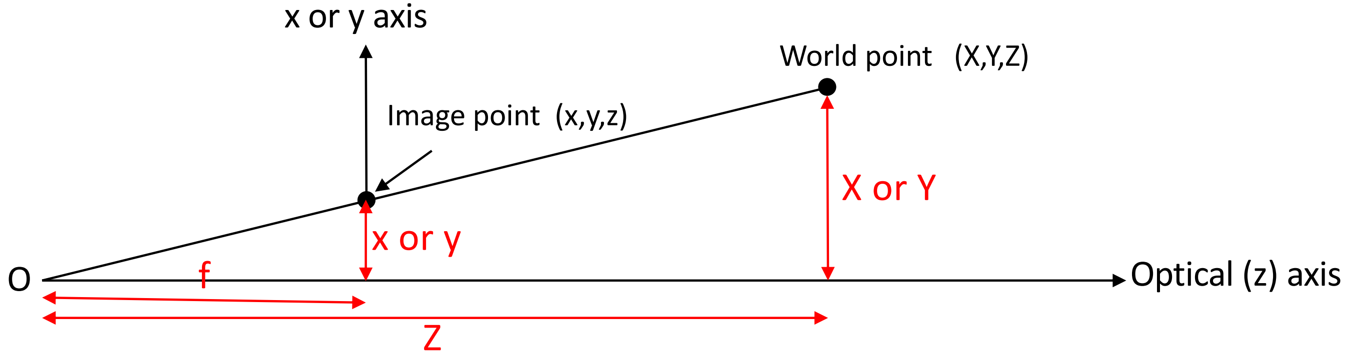
# 3D Geometry

## Perspective projections



# 3D Geometry

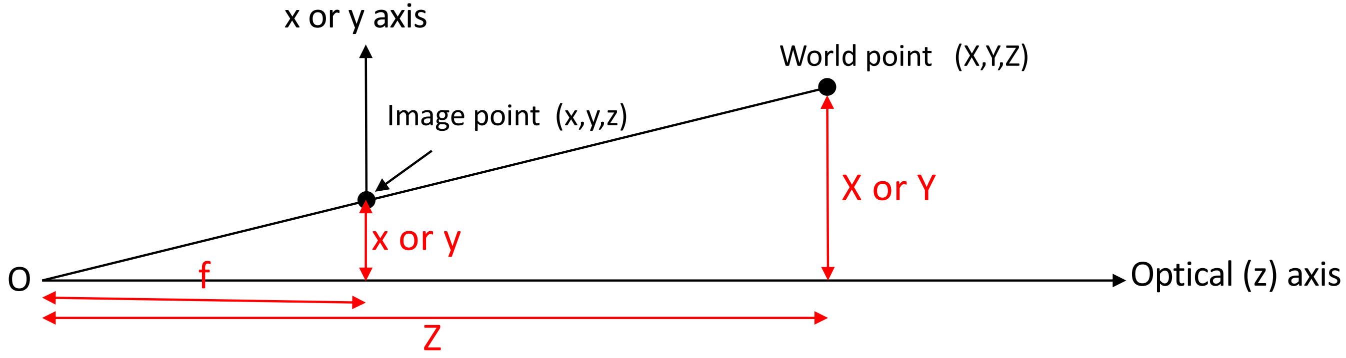
## Perspective projections




Using thales: X/x = Z/f and Y/y=Z/f
Thus: x=f X/Z, y= f Y/Z, z=f.

# 3D Geometry

## Perspective projections




Using thales: X/x = Z/f and Y/y=Z/f

Thus: x=f X/Z, y= f Y/C, z=f.

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \\ Z/f \end{bmatrix} \sim \begin{bmatrix} fX/Z \\ fY/Z \\ f \\ 1 \end{bmatrix}$$

Perspective projection matrix

# 3D Geometry

## Perspective projections

**Exercises**

1. Show that when the origin $O$ is in the image plane along the optical axis, the perspective projection matrix becomes:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/f & 1 \end{bmatrix}$$

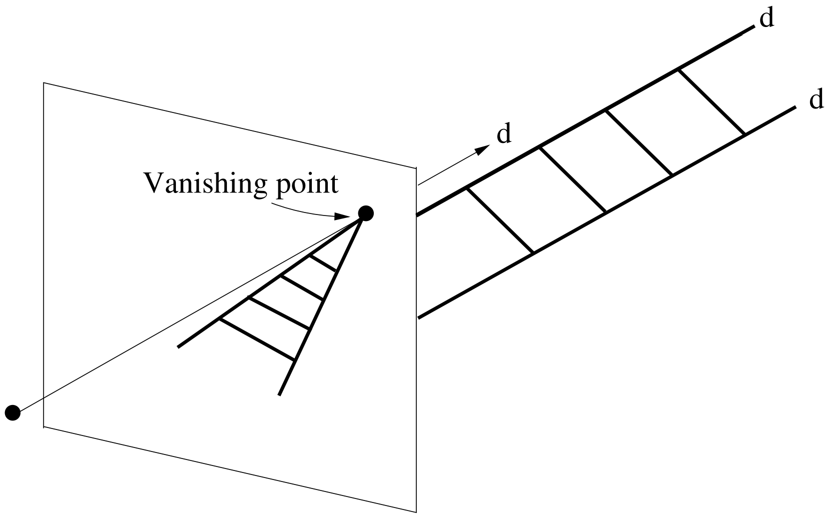
2. What is then the link between the orthographic and perspective projections ?

3. Circle projection: assume a circle of radius $R$ located in plane parallel to the image plane, at a distance $Z$, and such that its center is on the optical axis. Show that its projection is a circle of radius fR/(Z+f).

4. The circle is moved in a direction that belongs to the image plane, what becomes its projection ?

# 3D Geometry

## Perspective projections




Parallel lines intersect at infinity at the same location which, once projected, defines a vanishing point.

Université Grenoble Alpes

# 3D Geometry

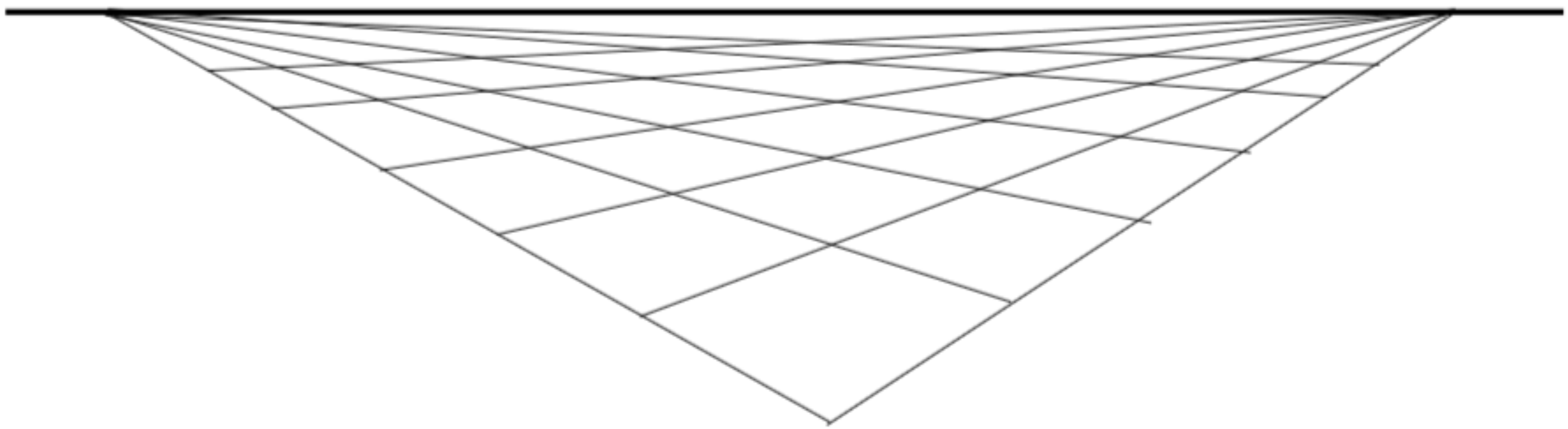
## Perspective projections

*La cène, Leonardo Da Vinci*

Vanishing points in perspective paintings

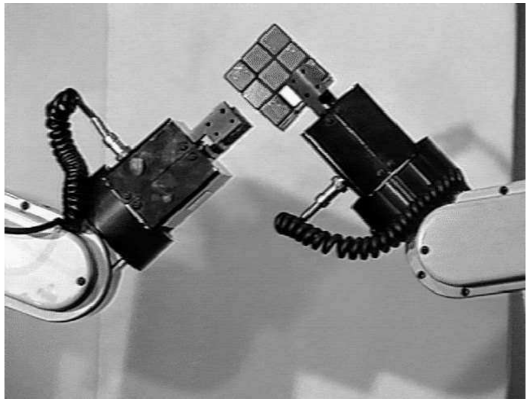
# 3D Geometry

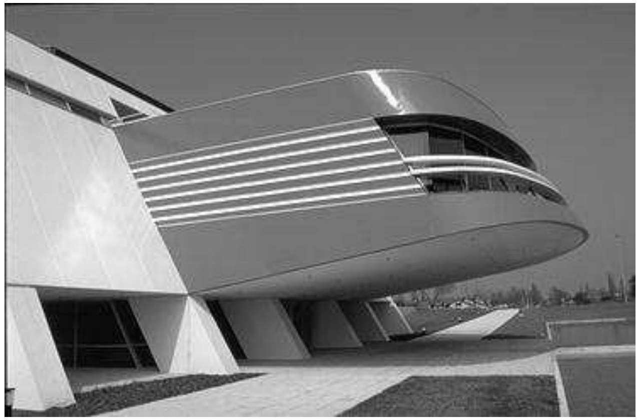
## Perspective projections



For lines in a plane, vanishing points define a line called horizon line.

# Single View Geometry

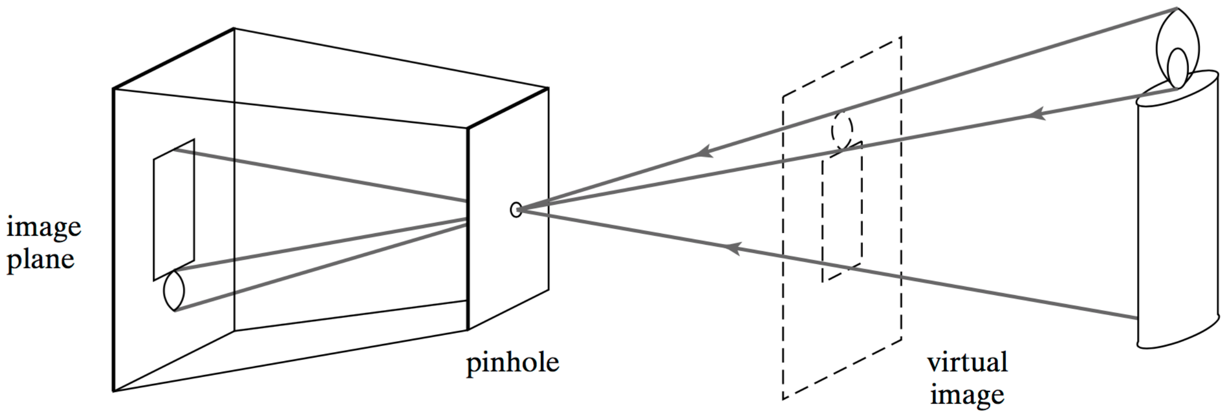




Camera Model: Projection parallel or perspective ?

# Single View Geometry

In practice the mostly used camera model is the pinhole model:




*Forsyth & Ponce, computer vision book*

And the full transformation from 3D to 2D is modeled as a projective transformation that includes a perspective projection.

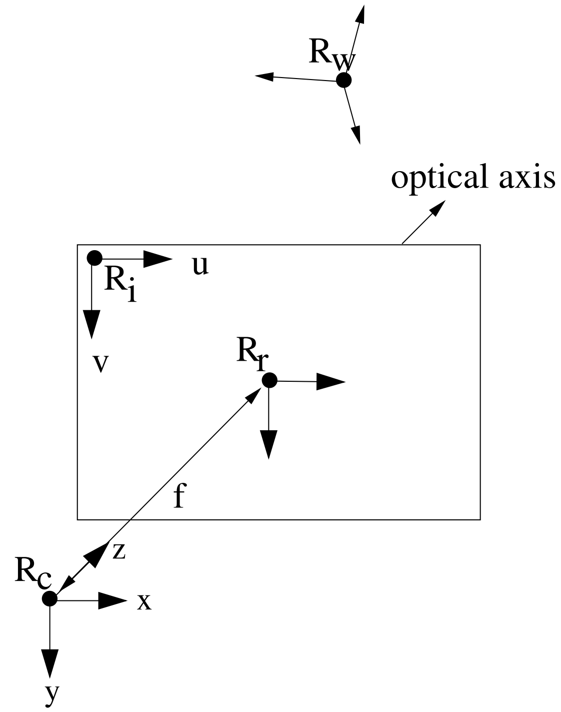
# Single View Geometry

The full transformation is composed of:

1. A rigid transformation between the world coordinate frame and the camera coordinate frame: Rw -> Rc.

2. A perspective projection into the retinal plane: Rc -> Rr.

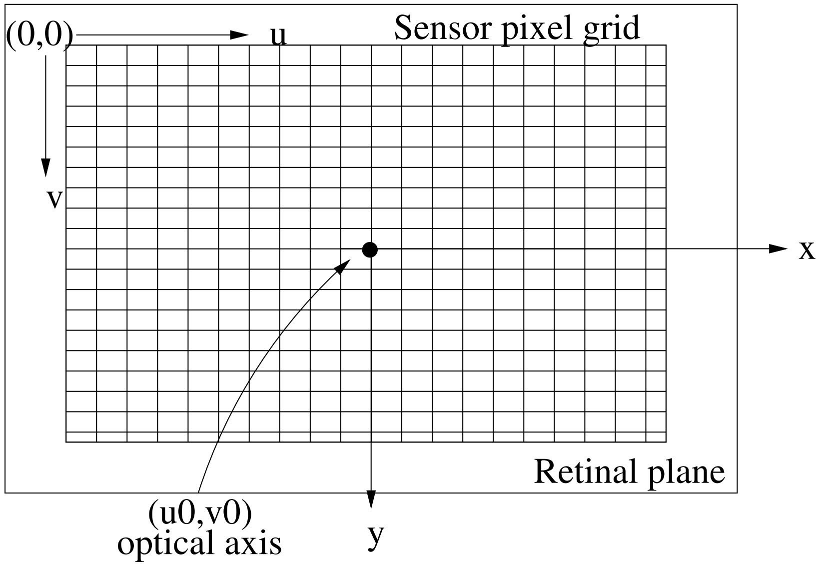
3. A 2D transformation from retinal coordinates to image pixel coordinates: Rr -> Ri



# Single View Geometry

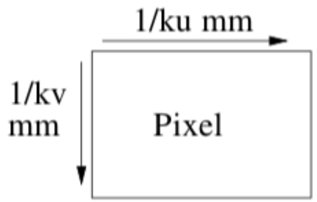
Retinal to image plane transformation:

$$\begin{pmatrix} u \\ v \\ f \\ 1 \end{pmatrix} = \begin{pmatrix} k_u & 0 & 0 & u_0 \\ 0 & k_v & 0 & v_0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ f \\ 1 \end{pmatrix}$$




where:

1. $k_u$ et $k_v$ are the scale factors in pixels/mm.




2. $(u_0, v_0)$ are the coordinates, in pixels, of the optical axis intersection with the retinal plane.

# Single View Geometry

The global transformation:

$$\begin{pmatrix} wu \\ wv \\ wf \\ w \end{pmatrix} = \begin{pmatrix} k_u & 0 & 0 & u_0 \\ 0 & k_v & 0 & v_0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 0 \end{pmatrix} \cdot \begin{pmatrix} & R & & T \\ & & & \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

# Single View Geometry

The global transformation:

$$\begin{pmatrix} wu \\ wv \\ wf \\ w \end{pmatrix} = \begin{pmatrix} k_u & 0 & 0 & u_0 \\ 0 & k_v & 0 & v_0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 0 \end{pmatrix} \cdot \begin{pmatrix} & R & & T \\ & & & \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} wu \\ wv \\ w \end{pmatrix} = \begin{pmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 0 \end{pmatrix} \cdot \begin{pmatrix} & R & & T \\ & & & \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Université Grenoble Alpes

# Single View Geometry

The global transformation:

$$\begin{pmatrix} wu \\ wv \\ w \end{pmatrix} = \begin{pmatrix} k_u f & 0 & u_0 \\ 0 & k_v f & v_0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} R & T \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

When pixels on the sensor are not rectangular:

$$\begin{pmatrix} wu \\ wv \\ w \end{pmatrix} = \begin{pmatrix} k_u f & c \neq 0 & u_0 \\ 0 & k_v f & v_0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} R & T \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

# Single View Geometry

The global transformation considered in practice (no distinction between scale factors and the focal length) :

$$M \sim \begin{pmatrix} \alpha_u & c & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} & R & & T & \end{pmatrix} = K \cdot \begin{pmatrix} & R & & T & \end{pmatrix},$$

$$M \sim (K \cdot R \quad K \cdot T).$$

Where:

1. K is the 3x3 intrinsic parameter matrix, i.e. the camera intrinsics.
2. [R T] is the 3x4 extrinsic parameter matrix, i.e. the camera location.

A camera is therefore described by 11 parameters which corresponds the degree of freedom of a 3x4 projective matrix. The calibration of camera consists in estimating the matrix M and the camera parameters.

Université Grenoble Alpes

# Single View Geometry

**Exercises**

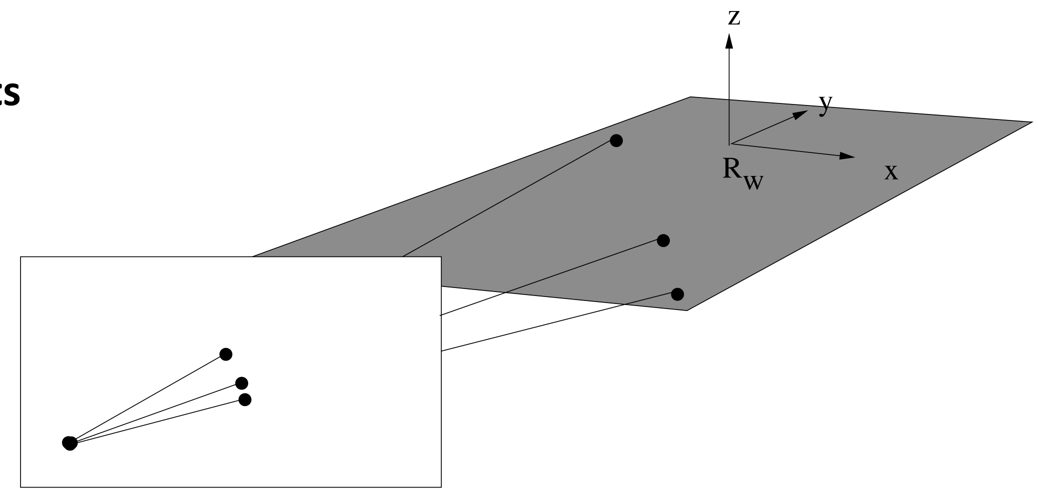
1. $M$ has 11 dof, how many 3D-2D correspondences are required to estimate $M$ ?
2. Denoting $\overline{M}$ the 3x3 matrix and $m$ the 3x1 vector such that $M \sim (\overline{M}\ m)$. Show that the location $C$ of the camera projection centre in the world coordinate frame is

$$C = -\overline{M}^{-1} \cdot m.$$

3. Given an estimation of $M$ by, e.g. calibration, how can the camera parameters be computed ?

# Single View Geometry

**Application: Coplanar points**




Assume, without loss of generality, that points belong to the plane z=0, then:

$$\begin{pmatrix} wu \\ wv \\ w \end{pmatrix} = \begin{pmatrix} \alpha_u & c & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} R_{11} & R_{12} & T_x \\ R_{21} & R_{22} & T_y \\ R_{31} & R_{32} & T_z \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

# Single View Geometry

Thus:

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \sim \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

The projection in this case is therefore a 2D projective transformation of the plane.

What can be deduced for the transformation between two perspectives projections of coplanar points ?

# Single View Geometry

**Coplanar points:** The orthographic case with the origin of the camera coordinate frame in the image plane:

$$\begin{pmatrix} wu \\ wv \\ w \end{pmatrix} = \begin{pmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 1 \end{pmatrix} \cdot \begin{pmatrix} & R & & T \\ & & & \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

With an infinite focal length and assuming the observed to be in the plane z=0 in the world coordinate frame:

$$\begin{pmatrix} wu \\ wv \\ w \end{pmatrix} = \begin{pmatrix} k_u & 0 & u_0 \\ 0 & k_v & v_0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} R_{11} & R_{12} & T_x \\ R_{21} & R_{22} & T_y \\ R_{31} & R_{32} & T_z \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix}$$

Thus:

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \sim \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & m_{33} \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix}$$

The projection in this case is therefore a 2D affine transformation of the plane.